Contents

Chapter 1

Numbers

1.1 Real numbers

Theorem 1 (Completeness axiom)**.** *Every nonempty set S of real numbers that is bounded from above has a supremum, i.e.* $\sup S$ *exists.*

Corollary 1 (Archimedean Property)**.** *If $a > 0$ and $b > 0$, then for some positive integer n, we have $na > b$.*

Proof. Define $S = \{na | n \in \mathbb{N}\}$.

(Proof by contradiction.) Assume this corollary is wrong, i.e. $\forall s_0 \in S : s_0 \leq b$. Thus, b is an upper bound of S and $\sup S \neq \infty$.

Let $s^* = \sup S \neq \infty$. Then, $s^* - a < s^*$ since $a > 0$.

Also, $s^* - a$ is not the upper bound of S. Thus, there exists $n \in \mathbb{N}$ s.t. $na > s^* - a$.

Then, we have:

$$(n+1)a = na + a > s^* - a + a = s^*$$

Contradiction. s^* is not upper bound of S anymore. □

Corollary 2 (Denseness of $\mathbb{Q}$)**.** *If $a,\ b \in \mathbb{Q}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.*

Proof. It suffices to show there exists $m, n \in \mathbb{Z}$ such that $a < \frac{m}{n} < b$.

Since $b - a > 0$, using Archimedean property, there exists $n \in \mathbb{N}$ such that $n(b-a) > 1$.

Then, $nb > na + 1$. Thus, there exists $m \in \mathbb{Z}$ s.t. $nb > na$ □

1.2 Proof by induction

Theorem 2. *Suppose that $A \subset \mathbb{N}$ is a set of natural numbers s.t.*

(a) $1 \in A$

(b) $n \in A$ *implies* $(n+1) \in A$.

Then $A = \mathbb{N}$.

1.3 sup, inf

Definition 1. *Suppose* $A \subset \mathbb{R}$ *is a set of real numbers.*

- $\sup A$ *is the lowest upper bound.*
- $\inf A$ *is the greatest lower bound.*

Key: Given $\epsilon > 0$, there exists $a \in A$ s.t. $a \leq \sup A < a + \epsilon$.

Chapter 2

Sequences

2.1 Defintion of limits

Definition 2 (Limit of a sequence)**.** *Given a sequence* (x_n)*.*

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \forall\epsilon > 0 : \exists N \in \mathbb{N} \ s.t. \ |x_n - x| < \epsilon \quad \forall n > N$$

Definition 3. *Given a sequence* (x_n)*.*

- $\lim_{n\to\infty} x_n = \infty \Leftrightarrow \forall M \in \mathbb{R} : \exists N \in \mathbb{N} \ s.t. \ x_n > M \quad \forall n > N$
- $\lim_{n\to\infty} x_n = -\infty \Leftrightarrow \forall M \in \mathbb{R} : \exists N \in \mathbb{N} \ s.t. \ x_n < M \quad \forall n > N$

2.1.1 Properties of limits

Key: $\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} |x_n - x| = 0$.

Proposition 1 (Uniqueness)**.** *Limit is unique.*

Proof. (Proof by contradiction.) Assume $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = x'$ s.t. $x \neq x'$.

By definition, $\forall\epsilon > 0 : \exists N, N' > 0$ s.t. $|x_n - x| < \epsilon/2 \quad \forall n > N$ and $|x_n - x'| < \epsilon/2 \quad \forall n > N'$.

Then, we have:

$$|x - x'| = |x - x_n + x_n - x'| \leq |x - x_n| + |x_n - x'| < \epsilon/2 + \epsilon/2 = \epsilon$$

By deifnition, $\lim_{n\to\infty} |x - x'| = 0$. So, $x = x'$. □

Theorem 3 (Boundness)**.** *If limit exists, then sequence* (x_n) *is bounded.*

Proof. Let $\lim_{n\to\infty} x_n = x$. Then, $\exists N > 0$ s.t. $|x_n - x| < 1 \quad \forall n > N$.

Thus, $\forall n > N : |x_n| = |x_n - x + x| \leq |x_n - x| + |x| < |x| + 1$, which is a constant.

Let $M = \max(|x_1|, \ldots, |x_N|, |x| + 1)$.

Then, we have $|x_n| \leq M \quad \forall n \in \mathbb{N}$, and the sequence (x_n) is bounded. □

Theorem 4 (Exchange the order of limits and algebraic operations)**.** *Given $(x_n), (y_n)$ are convergent sequences ($\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$), $c \in \mathbb{R}$.*

Then, sequences $(cx_n), (x_n + y_n), (x_n \cdot y_n)$ are convergent, and:

- $\lim_{n\to\infty} cx_n = cx$ *Choose $|x_n - x| < \epsilon/|c|$.*
- $\lim_{n\to\infty}(x_n + y_n) = x + y$ *Choose $|x_n - x| < \epsilon/2 \ \forall n > N_1$, $|y_n - y| < \epsilon/2 \ \forall n > N_2$.*
- $\lim_{n\to\infty}(x_y \cdot y_n) = xy$ *$\exists M > 0$ s.t. $|x_n|, |y_n| \leq M \ \forall n$. $|x_n y_n - xy| \leq |x_n||y_n - y| + |y||x_n - x|$.*

Theorem 5 (Preserve monotonicity)**.** *If $(x_n), (y_n)$ are convergent and $x_n \leq y_n \quad \forall n \in \mathbb{N}$, then:*

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$$

Proof. By defintion, $\forall \epsilon > 0 : \exists N_1, N_2 > 0$ s.t. $|x_n - x| < \epsilon/2 \quad \forall n > N_1$ and $|y_n - y| < \epsilon/2 \quad \forall n > N_2$. Thus, $y - \epsilon/2 < y_n < y + \epsilon/2$.

Choose $N = \max(N_1, N_2)$. Then, we have $\forall n > N$:

$$x = x_n + x - x_n < y_n + \frac{\epsilon}{2} < y + \frac{\epsilon}{2} + \frac{\epsilon}{2} = y + \epsilon \quad \forall \epsilon > 0$$

i.e. $x \leq y$.

□

Theorem 6 (Squeeze theorem*)**.** *Let $(x_n), (y_n)$ be the convergent sequences with the same limit L. If a sequence (z_n) is such that*

$$x_n \leq z_n \leq y_n \quad \forall n \in \mathbb{N}$$

then (z_n) also converges to L.

2.2 limsup, liminf

Theorem 7. *Given a sequence (x_n). Then:*

$$y = \limsup_{n\to\infty} x_n$$

iff $y \in [-\infty, \infty]$ satisfies one of the following:

(1) $y \in (-\infty, \infty)$: $\forall \epsilon > 0$

(a) $\exists N \in \mathbb{N}$ s.t. $x_n < y + \epsilon \quad \forall n > N$

(b) $\forall N \in \mathbb{N} : \exists n > N$ s.t. $x_n > y - \epsilon$

(2) $y = \infty$: $\forall M \in \mathbb{R} : \exists n \in \mathbb{N}$ s.t. $x_n > M$, i.e. (x_n) is not bounded from above

(3) $y = -\infty$: $\forall m \in \mathbb{R} : \exists N \in \mathbb{N}$ s.t. $x_n < m \quad \forall n > N$, i.e. $x_n \to -\infty$ as $n \to \infty$

Similarly:

$$z = \liminf_{n\to\infty} x_n$$

iff $z \in [-\infty, \infty]$ satisfies one of the following:

(1) $z \in (-\infty, \infty)$: $\forall \epsilon > 0$

(a) $\exists N \in \mathbb{N} : x_n > z - \epsilon \quad \forall n > N$

(b) $\forall N \in \mathbb{N} : \exists n > N$ s.t. $x_n < z + \epsilon$

(2) $x = \infty$: $\forall M \in \mathbb{R} : \exists N \in \mathbb{N}$ s.t. $x_n > M \quad \forall n > N$, i.e. $x_n \to \infty$ as $n \to \infty$

(3) $x = -\infty$: $\forall m \in \mathbb{R} : \exists n \in \mathbb{N}$ s.t. $x_n < m$, i.e. (x_n) is not bounded from below

Theorem 8.

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \limsup x_n = \liminf x_n = x$$

Proof. For $\Leftarrow$: Given $\limsup x_n = \liminf x_n = x$.

Note that $\lim_{n\to\infty} \sup\{x_k : k \geq n\} = x$ decreases and $\lim_{n\to\infty} \inf\{x_k : k \geq n\} = x$ increases as $n \to \infty$.

Let $z_n = \inf\{x_k : k \geq n\}$, which is monotone increasing and bounded above. Let $y_n = \sup\{x_k : k \geq n\}$.

Then, we have: $x - \epsilon < z_n \leq x_n \leq y_n < x + \epsilon \quad \forall \epsilon > 0$.

For $\Rightarrow$: Given $\lim x_n = x$.

By definition, $\forall \epsilon > 0 : \exists N > 0$ s.t. $|x_n - x| < \epsilon$, i.e. $x - \epsilon < x_n < x + \epsilon \quad \forall n > N$.

Then, we have: $x - \epsilon < z_n \leq x_n \leq y_n < x + \epsilon$. Thus, $\lim z_n = \lim y_n = x = \limsup x_n = \liminf x_n$. □

Corollary 3.

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \limsup |x_n - x| = 0$$

Key: $\liminf x_n \leq \limsup x_n$.

Proof. Note that $\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} |x_n - x| = 0 = \limsup |x_n - x|$.

Conversely, if $\limsup |x_n - x| = 0$, we have:

$$0 \leq \liminf |x_n - x| \leq \limsup |x_n - x| = 0$$

i.e. $\liminf |x_n - x| = \limsup |x_n - x| = 0$ and thus $\lim x_n = x$. □

Problem: Prove $\liminf x_n = -\limsup(-x_n)$ for any sequence (x_n).

Proof. Let $x_N = \inf\{x_n : n > N\} = -\sup\{-x_n : n > N\}$. Then, we have:

$$\lim_{N\to\infty} x_N = \liminf x_n = -\limsup(-x_n)$$

□

2.3 Monotone sequences*

Theorem 9. *A monotone sequence converges iff it is bounded.*

- *If (x_n) is monotone increasing, bounded, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.*
- *If (x_n) is monotone decreasing, bounded, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$.*

2.4 Cauchy

Definition 4 (Cauchy sequence)**.** *Given (x_n), we say (x_n) is a Cauchy sequence if $\forall \epsilon > 0 : \exists N > 0$ s.t. $\forall n, m > N$ we have $|x_n - x_m| < \epsilon$.*

Theorem 10. *(x_n) converges $\Leftrightarrow$ x_n is a Cauchy sequence.*

Proof. For $\Rightarrow$: Given $\lim_{n\to\infty} x_n = x$. By definition, $\forall \epsilon > 0 : \exists N > 0$ s.t. $|x_n - x| < \epsilon/2 \quad \forall n > N$.

Then, we have $\forall n, m > N$:

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

For $\Leftarrow$: Given (x_n) is Cauchy.

Choose $\epsilon = 1$. Then, $\exists N_1 > 0$ s.t. $\forall n, m > N_1$: $|x_n - x_m| < 1$.

Thus, if $m > N_1$:

$$|x_m| = |x_m - x_{N_1+1} + x_{N_1+1}| \leq |x_m - x_{N_1+1}| + |x_{N_1+1}| \leq 1 + |x_{N_1+1}|$$

i.e. (x_n) is bounded. Thus, its limsup and liminf exist and are finite.

Let $\limsup_{n\to\infty} x_n = a$, and we prove $\lim_{n\to\infty} x_n = a$.

(1) By property of limsup, $\forall \epsilon > 0 : \exists N_\epsilon > 0$ s.t. $x_n \leq a + \epsilon/2 \quad \forall n > N_\epsilon$

(2) By definition of Cauchy sequence, $\forall \epsilon > 0 : \exists N' > 0$ s.t. $|x_n - x_m| < \epsilon/2 \quad \forall n, m > N'$

(3) By property of limsup again, $\forall \epsilon > 0 : \exists n^* > \max(N_\epsilon, N')$ s.t. $x_{n^*} > a - \epsilon/2$

Combining (1) and (3), we have: $|x_{n^*} - a| < \epsilon/2 \quad \forall n^* > \max(N_\epsilon, N')$

Combining (2) and above, we have: $|x_n - a| = |x_n - x_{n^*} + x_{n^*} - a| \leq |x_n - x_{n^*}| + |x_{n^*} - a| < \epsilon/2 + \epsilon/2 = \epsilon$

Thus, $\lim_{n\to\infty} x_n = a$ by defintion. □

Problem: [Exercise 10.6]

(a) Let (s_n) be a sequence s.t. $|s_{n+1} - s_n| < \frac{1}{2^n} \quad \forall n \in \mathbb{N}$. Prove (s_n) is Cauchy.

Proof. Choose $m > n > N > 0$:

$$|s_n - s_m| \leq |s_n - s_{n+1}| + |s_{n+1} - s_{n+2}| + \cdots + |s_{m-1} - s_m| \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \leq 2^{-N} < \epsilon$$

for large enough $N >> 1$. □

(b) Does it hold if $|s_{n+1} - s_n| < \frac{1}{n} \quad \forall n \in \mathbb{N}$?

Solution: No. Consider $s_n = \sum_{k=1}^{n} \frac{1}{k} \to \infty$ as $n \to \infty$. Thus, the sequence (s_n) is not bounded from above, and (s_n) cannot be Cauchy. However, $s_{n+1} - s_n = \frac{1}{n+1} < \frac{1}{n}$ is satisfied.

2.5 Subsequences

Proposition 2. *Every subsequence of a a convergent sequence converges to the limit of the sequence.*

Proof. Given a sequence (x_n) and $\lim_{n\to\infty} x_n = x$. Fix $\epsilon > 0$.

By definition, there exists $N > 0$ s.t. $|x_n - x| < \epsilon \quad \forall n > N$.

Observe a subsequence (x_{n_k}). Pick $K > 0$ such that $n_k > N$ for $k > K$. Then:

$$|x_{n_k} - x| < \epsilon \quad \forall k > K$$

and (x_{n_k}) converges to x.

□

2.6 B-W theorem

Theorem 11 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Key: Every bounded sequence must have a monotonic subsequence.

Proof. We construct a subsequence converging to $y = \limsup x_n$.

$\forall \epsilon > 0$:

(a) $\exists N \in \mathbb{N}$ s.t. $x_n < y + \epsilon$.

(b) $\forall N \in \mathbb{N} : \exists n > N$ s.t. $x_n > y + \epsilon$.

- Choose $\epsilon_1 = \frac{1}{2}$. Then $\exists N_1$ s.t. $x_n < y + \epsilon_1 \quad \forall n > N$. Also, there exists n_1 s.t. $x_n > y - \epsilon_1$.

  So, $|x_{n_1} - y| < \epsilon_1$.

- Choose $\epsilon_2 = \frac{1}{2^2}$. Then $\exists N_2$ s.t. $x_n < y + \epsilon_2 \quad \forall n > N$. Also, there exists $n_2 > \max(N_1, N_2)$ s.t. $x_n > y - \epsilon_2$.

  So, $|x_{n_2} - y| < \epsilon_2$.

- Iteratively, for $\epsilon_k = \frac{1}{2^k}$: $\exists N_k$ s.t. $x_n < y + \epsilon_k \quad \forall n > N$. Also, $\exists n_k > \max(N_k, N_{k-1})$ s.t. $x_{n_k} > y - \epsilon_k$. Thus, $|x_{n_k} - y| < \epsilon_k$.

Therefore, $x_{n_k} \to y$ and (x_{n_k}) converges. □

Chapter 3

Series

"Series is infinite sum of real numbers"

Definition 5 (Series)**.** *Given a sequence* (a_n)*. The series* $\sum_{n=1}^{\infty} a_n$ *converges to a sum* $S \in \mathbb{R}$ *if the sequence* (S_n) *of partial sums* $S_n = \sum_{k=1}^{n} a_k$ *converges to* S *as* $n \to \infty$*. Otherwise, the series diverges.*

Key: $\sum a_n$ converges implies $\lim a_n = 0$.

Theorem 12 (Cauchy condition for series)**.** *The series* $\sum_{n=1}^{\infty} a_n$ *converges iff* $\forall \epsilon > 0 : \exists N \in \mathbb{N}$ *s.t. the tail of the series can be arbitrarily small:*

$$\sum_{k=m+1}^{n} |a_k| < \epsilon \quad \forall n > m > N$$

Proof. The series converges iff the sequence of partial sums (S_n) is Cauchy:

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} \text{ s.t. } |S_n - S_m| = \sum_{k=m+1}^{n} |a_k| < \epsilon \quad \forall n > m > N$$

□

Corollary 4. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$*.*

Proof. If the series converges, then it is Cauchy. Choose $m = n - 1$:

$$|S_n - S_{n-1}| = |a_n| < \epsilon \quad \forall \epsilon > 0, n > n - 1 > N$$

□

Definition 6 (Absolute convergence)**.** *The series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

Key: Absolute convergence implies convergence.

3.1 Three tests

Theorem 13 (Comparison test)**.** *Suppose that* $b_n \geq 0$ *and* $\sum_{n=1}^{\infty} b_n$ *converges. If* $|a_n| \leq b_n$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Since $\sum b_n$ converges, it satisfies the Cauchy condition. Since:

$$\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} b_k$$

the series $\sum |a_n|$ also satisfies the Cauchy condition. Therefore $\sum a_n$ converges too. □

Theorem 14 (Ratio test)**.** *Given a sequence* (a_n)*.*

- *It converges absolutely if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$*.*
- *It diverges if* $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$*.*

Theorem 15 (Root test)**.** *Given a sequence* (a_n) *and* $r = \limsup_{n\to\infty} |a_n|^{1/n}$*.*
Then $\sum_{n=1}^{\infty} a_n$ *converges absolutely if* $0 \leq r < 1$ *and diverges if* $1 < r \leq \infty$*.*

Theorem 16 (Ratio implies Root)**.** *Given a sequence* (a_n)*.*

- *If* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$*, then* $\limsup_{n\to\infty} |a_n|^{1/n} < 1$*.*
- *If* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$*, then* $\limsup_{n\to\infty} |a_n|^{1/n} > 1$*.*

Chapter 4

Limit of a function

Boundedness of a function with a limit. $f : A \to \mathbb{R}$, a is an accumulation point of A, $\lim_{x \to a} f(x) = f(a)$. Then $\exists \delta > 0$ and $K > 0$ s.t. if $|x - a| < \delta$ then $|g(x)| \leq K$.

Definition 7 (Accumulation point)**.** *Given a set A. Then c is called an accumulation point of A if for any $\delta > 0$, there exists $a \neq c$ s.t. $|a - c| < \delta$ and $a \in A$.*

Definition 8 (Limit of a function)**.** *Let $f : A \to \mathbb{R}$, where $A \subset \mathbb{R}$, and suppose that $c \in \mathbb{R}$ is an accumulation point of A. Then:*

$$\lim_{x \to c} f(x) = L \Leftrightarrow \forall \epsilon > 0 : \ \exists \delta > 0 \ s.t. \ if \ |x - c| < \delta, x \in A \ then \ |f(x) - L| < \epsilon \tag{4.1}$$

4.0.1 Classes of sets

Definition 9 (Open set)**.** *A set $A \subseteq \mathbb{R}$ is open, if for any point $a \in A$, there exists an interval satisfying $a \in (b, c)$ and $(b, c) \subseteq A$.*

Proposition 3. *A is open set iff for any $a \in A$, there exists $\delta > 0$ s.t. for any point c satisfying $d(a, c) < \delta$, we have $c \in A$.*

Definition 10 (Interior point)**.** *Given a set $A \subseteq \mathbb{R}$, $a \in A$ is called interior point of A if there exists an interval satisfying $a \in (b, c)$ and $(b, c) \in A$.*

Definition 11 (Accumulation point)**.** *Given a set $A \subseteq \mathbb{R}$, $a \in A$ is called an accumulation point of A if for any interval satisfying $a \in (b, c)$, we always have $(b, c) \cap A \backslash \{a\} \neq \varnothing$.*

Proposition 4. *Given a set A*

- *a is an interior point of A iff there exists $\epsilon > 0$ s.t. for any point c satisfying $d(a, c) < \epsilon$, we have $c \in A$.*
- *a is an accumulation point of A iff for any $\epsilon > 0$, there exists a point c s.t. $d(a, c) < \epsilon$, $c \in A$, and $c \neq a$.*

Chapter 5

Continuous function

Definition 12 (Continuous function)**.** $f : A \to \mathbb{R}, A \subseteq R, c \in A$.
f is continuous at c if $\forall \epsilon > 0 : \exists \delta > 0$ s.t. if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.

Alternatively, f is continuous at c if $\lim_{x \to c} f(x) = f(c)$.

5.1 Uniform continuity

Definition 13. *$f : A \to \mathbb{R}$, f is uniformly continuous function in A if $\forall \epsilon > 0 : \exists \delta_\epsilon > 0$ s.t. for $x, y \in A$ if $|x - y| < \delta_\epsilon$ then $|f(x) - f(y)| < \epsilon$.*

Key: Continuous function on a closed, bounded (compact) set is uniform continuous!

Proposition 5. *f is not uniform continuous $\Leftrightarrow \exists \epsilon_0 > 0$ and two sequences $(x_n), (y_n)$ s.t.*

$$\lim_{n \to \infty} |x_n - y_n| = 0 \text{ but } |f(x_n) - f(y_n)| > \epsilon_0 \quad \forall n$$

Definition 14 (Lipschitz function)**.** *f is L-Lipschitz if*

$$\forall x, y : |f(x) - f(y)| \leq L \cdot |x - y|$$

Then f is uniform continuous.

Problem: [Mid-Term-1] Let $f(x), g(x)$ be continuous functions on $[0, 1]$ s.t. $f(0) < g(0)$ and $f(1) > g(1)$. Let

$$E = \{x | x \in [0, 1], f(x) < g(x)\}$$

(a) Prove $\sup E < 1$. **Key:** By continuity, there exists $\delta > 0$ s.t. $\forall x \in [1 - \delta, 1] : f(x) > g(1)$.

(b) Prove $f(\sup E) = g(\sup E)$.

Theorem 17 (Compact means sequentially compact)**.** *A set A is compact (bounded and closed) if and only if A is sequentially compact, meaning that for any sequences (x_n) of A, there exists a subsequence (x_{n_k}) such that x_{n_k} converges to some point $a \in A$.*

Proof.
For $\Rightarrow$: Pick any $(x_n) \subseteq A$. Since A is bounded, (x_n) is a bounded sequence.

By Bolzano-Weierstrass theorem, there must exist a convergent subsequence (x_{n_k}). Let $\lim_{k\to\infty} x_{n_k} = a$.

We show that $a \in A$. (Proof by contradiction.)

Assume $a \in A^c$. Since A^c is open, $\exists \epsilon^* > 0$ s.t. $(a-\epsilon^*, a+\epsilon^*) \in A^c \Leftrightarrow (a-\epsilon^*, a+\epsilon^*) \cap A = \varnothing$.

Then, $(a-\epsilon^*, a+\epsilon^*) \cap \{x_{n_k}\} = \varnothing$ and $|x_{n_k} - a| > \epsilon^*$. Contradiction.

For $\Leftarrow$: We are given a sequentially compact set A. (Proof by contradiction.)

Assume A is not compact. Then A is either unbounded or open.

Assume A is unbounded. Then, we can construct a sequence (x_n) in A that diverges ($x_n \to \infty$ as $n \to \infty$), meaning it has no convergent subsequence. Contradiction.

Now, assume A is open. Then, there exist a sequence (x_n) in A s.t. it converges a point $a \in A^c$.

Thus, every subsequence (x_{n_k}) of this sequence also converges to a.

However, every sequence in A has a convergent subsequence with its limit in A. Contradiction.

Therefore, A must be bounded and closed, meaning it is compact. □

Theorem 18. *If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then $f(K)$ is compact.*

Theorem 19 (Theorem 4.4)**.** *If $f : K \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, then f is uniformly continuous on K.*

Proof. Suppose for contradiction that f is not uniformly continuous on K. Then, by reverse definition of uniform continuity, there exists $\epsilon_0 > 0$ and sequences (x_n), (y_n) in K such that

$$\lim_{n\to\infty} |x_n - y_n| = 0 \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \epsilon_0 \text{ for every } n \in \mathbb{N}.$$

Since K is compact, there is a convergent subsequence (x_{n_i}) of (x_n) such that $lim_{i\to\infty} x_{n_i} = x \in K$.

Moreover, since $(x_n - y_n) \to 0$ as $n \to \infty$, it follows that

$$\lim_{i\to\infty} y_{n_i} = \lim_{i\to\infty} [x_{n_i} - (x_{n_i} - y_{n_i})] = \lim_{i\to\infty} x_{n_i} - \lim_{i\to\infty} (x_{n_i} - y_{n_i}) = x,$$

so (y_{n_i}) also converges to x. Then, since f is continuous on K,

$$\lim_{i\to\infty} |f(x_{n_i}) - f(y_{n_i})| = \left|\lim_{i\to\infty} f(x_{n_i}) - \lim_{i\to\infty} f(y_{n_i})\right| = |f(x) - f(x)| = 0,$$

but this contradicts the non-uniform continuity condition $|f(x_{n_i}) - f(y_{n_i})| \geq \epsilon_0$. □

Theorem 20 (Weierstrass extreme value theorem)**.** *If $f : A \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is compact, then f is bounded on A and f attains its maximum and minimum values on A.*

5.2 Intermediate value theorem

Lemma 1 (Intermediate value)**.** *Given a continuous function* $f : [a, b] \to \mathbb{R}$*. If* $f(a) < 0$ *and* $f(b) > 0$*, or* $f(a) > 0$ *and* $f(b) < 0$*, then there exists* $c \in (a, b)$ *s.t.* $f(c) = 0$*.*

Proof. Assume $f(a) < 0$ and $f(b) > 0$.

Let $E = \{x \in [a, b] : f(x) < 0\}$.

Then, $a \in E$ and E is a nonempty set bounded above by b. By completeness axiom, $\sup E$ exists.

Let $c = \sup E$. We show that $f(c) = 0$.

By definition of continuity, $\forall \epsilon > 0 : \exists \delta > 0$ s.t. if $|x - c| < \delta$ then:

$$|f(x) - f(c)| < \epsilon = \frac{|f(c)|}{2}$$

(Proof by contradiction.)

Assume $f(c) < 0$: $c \neq b$. For all $x \in [a, b]$ s.t. $|x - c| < \delta$:

$$f(x) = f(c) + f(x) - f(c) < f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} < 0$$

i.e. $x \in E$ for any $x \in [c - \delta, c + \delta] \cap [a, b]$. Since $c < b$, there is $x^* = c + \delta \in E$ s.t. $x^* > c$. Contradiction.

(Note that we use denseness of $\mathbb{Q}$ here to show $c < c + \delta < b$ exists.)

Assume $f(c) > 0$: $c \neq a$. For all $x \in [a, b]$ s.t. $|x - c| < \delta$:

$$f(x) = f(c) + f(x) - f(c) < f(c) + \frac{f(c)}{2} > 0$$

i.e. $x \notin E$ for any $x \in [c - \delta, c + \delta] \cap [a, b]$. Then, $\forall x \in [c - \delta, c] : f(x) > 0$ and $x \notin E$.

Thus, $e < c - \delta \quad \forall e \in E$, i.e. $c - \delta$ is a lower upper bound than c. Contradiction.

□

Chapter 6

Differentiable function

6.1 Derivative

Definition 15. $f : (a, b) \to \mathbb{R}, c \in (a, b)$.

We say f is differentiable at point c if $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.

$$f'(c) \text{ exists } \Leftrightarrow f'(c^+) = f'(c^-)$$

6.1.1 Properties

Theorem 21 (Continuity)**.** *If f is differentiable at c, then f is continuous at c.*

Proof.

$$\lim_{x\to c} f(x) - f(c) = \lim_{h\to c} f(c+h) - f(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot h = f'(c) \cdot 0 = 0$$

□

Theorem 22 (Algebraic property)**.** *f, g are differentiable at c.*

1. $(f \pm g)'(c) = f'(c) \pm g'(c)$
2. $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$ *Key :* $\lim_{h\to 0} \frac{[f(c+h)-f(c)]g(c+h)-f(c)[g(c+h)-g(c)]}{h}$
3. *if* $g(c) \neq 0$*:* $(\frac{f}{g})'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{g^2(c)}$

Theorem 23 (Chain rule)**.** *f, g, g is differentiable at c, f differentiable at $g(c)$. Then, $f \circ g$ is differentiable at c, where $(f(g(c)))' = f'(g(c)) \cdot g'$.*

Proof.

$$\lim_{h\to 0} \frac{f(g(c+h)) - f(g(c))}{h} = \lim_{h\to 0} \frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} \frac{g(c+h) - g(c)}{h} = f'(g(c)) \cdot g'(c)$$

□

6.2 Mean value theorem

Proposition 6. *If x_0 is the max or min point on (a, b), then $f'(x_0) = 0$.*

Proof.
$$f'(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h \geq 0} \leq 0 = \lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h \leq 0} \geq 0 = 0$$
□

Theorem 24 (Rolle)**.** *f is continuous on $[a, b]$, differentiable on (a, b).*
Given $f(a) = f(b)$.
$$\exists c \in (a, b) : f'(c) = 0$$

Proof. Since f is a continuous function on a closed interval $[a, b]$, f attains its max/min values on $[a, b]$ (Weierstrass extreme value theorem).

If a, b are max/min, pick c to be min/max. Else, if a, b are not max/min, pick c to be max. □

Theorem 25 (Mean value)**.** *f is continuous on $[a, b]$, differentiable on (a, b).*
$$\exists c \in (a, b) : f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define $g(x) = f(x) - f(a) - \left[\frac{f(b)-f(a)}{b-a}\right](x - a)$, which is also cont. on $[a, b]$, diff. on (a, b):
$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$
where $g(a) = 0 = g(b)$. By Rolle's theorem, $\exists c \in (a, b) : g'(c) = 0$. □

Corollary 5. *$f : (a, b) \to \mathbb{R}$ is differentiable on (a, b), $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant on (a, b).*

Key: fix $x_0 \in (a, b)$. By MVT, $\forall x \in (a, b)$ s.t. $x \neq x_0 : \exists c$ between x and x_0 s.t. $f'(c) = \frac{f(x)-f(x_0)}{x-x_0} := 0$.

Corollary 6. *$f,\ g : (a, b) \to \mathbb{R}$ is differentiable on (a, b), $f'(x) = g'(x) \quad \forall x \in (a, b)$.*
Then, $f(x) = g(x) + C$ for some constant C.

Key: $(f - g)'(x) = 0 \quad \forall x \in (a, b)$. Then, $f - g$ is a constant function on (a, b).

Corollary 7. *$f : (a, b) \to \mathbb{R}$ is differentiable on (a, b). f is increasing on (a, b) iff $f'(x) \geq 0 \quad \forall x \in (a, b)$.*

Proof. $\Rightarrow$: given f is increasing, consider any $x \in (a, b)$: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \geq 0$.

$\Leftarrow$: given $f' \geq 0$, consider $a < x < y < b$. By MVT, $\exists c \in (x, y) : f'(c) = \frac{f(y)-f(x)}{y-x} \geq 0$, i.e. $f(y) \geq f(x)$. □

6.3 Inverse function theorem

Proposition 7. *$f : A \to \mathbb{R}$ is one-to-one on $A \subset \mathbb{R}$. Assume its inverse $f^-1 : B \to \mathbb{R}$ exists.*

Assume f is differentiable at $c \in A$, f^{-1} is differentiable at $f(c) \in B$.

Then, given $f'(c) \neq 0$, we have:

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)} \tag{6.1}$$

Proof. By definition, $f^{-1}(f(x)) = x$. Take derivative on both sides and apply chain rule. □

Theorem 26 (Inverse function). *$f : A \to \mathbb{R}$, c is an interior point, $f'(x)$ is continuous on A.*

If $f'(c) \neq 0$, then one can find $\delta > 0$ s.t.

(1) f is one-to-one on $(c - \delta, c + \delta) \Rightarrow f^{-1}$ exists on $(c - \delta, c + \delta)$

(2) f^{-1} is differentiable on $f((c - \delta, c + \delta)) \Rightarrow (f^{-1})'(f(c)) = \frac{1}{f'(c)}$

Proof. WLOG assume $f'(c) > 0$. Then, since f' is continuous, $f'(x) > 0$ for $x \in (c - \delta, c + \delta)$, i.e. f is increasing, one-to-one on $x \in (c - \delta, c + \delta)$.

We prove that f^{-1} is continuous on $a \in f((c - \delta, c + \delta))$ by showing $\lim_{y \to a} f^{-1}(y) = f^{-1}(a)$.

$\forall \epsilon > 0 : \exists \delta = \min\{a - f(f^{-1}(a) - \epsilon), f(f^{-1}(a) + \epsilon) - a\}$ s.t. if $|y - a| < \delta$ then $|f^{-1}(y) - f^{-1}(a)| < \epsilon$.

Consider:

$$\lim_{y \to a} \frac{f^{-1}(y) - f^{-1}(a)}{y - a} = \lim_{h \to 0} \frac{f^{-1}(a + h) - f^{-1}(a)}{h} = \lim_{h \to 0} \frac{f^{-1}(a + h) - f^{-1}(a)}{f(f^{-1}(a + h)) - f(f^{-1}(a))} = \lim_{x \to c} \frac{x - c}{f(x) - f(c)} = \frac{1}{f'(c)} \tag{6.2}$$

□

6.4 L'Hospital's rule

Theorem 27 (Cauchy mean value). *f, g are continuous on $[a, b]$, differentiable on (a, b).*

$$\exists c \in (a, b) : \quad f'(c) \cdot [g(b) - g(a)] = g'(c) \cdot [f(b) - f(a)]$$

Proof. Define $h(x) = [f(x) - f(a)][g(b) - g(a)] - [f(b) - f(a)][g(x) - g(a)]$, where $h(a) = 0 = h(b)$.

By Rolle, $\exists c \in (a, b) : h'(x) = 0$. □

Theorem 28 (L'Hospital's rule). *f, $g : (a, b) \to \mathbb{R}$ are differentiable on (a, b), where $g'(x) \neq 0 \quad \forall x \in (a, b)$ and $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$.*

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a^+} \frac{f(x)}{g(x)} = L$$

Proof. Extend to $f, g : [a, b) \to \mathbb{R}$ by setting up $f(a) = g(a) = 0$.

Fix $x \in (a, b)$. By MVT, $\exists c \in (a, x)$ s.t. $g'(c) = \frac{g(x)-g(a)}{x-a}$.

Note that $g(x) = g(x) - g(a) = g'(c) \cdot (x - a) \neq 0$ by given conditions, so $g(x) \neq 0 \quad \forall x \in (a, b)$.

By Cauchy MVT, $\exists c \in (a, b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f(x)}{g(x)}$ is well-defined. As $x \to a^+$, $c \to a^+$. □

6.5 Taylor's theorem

Definition 16 (Taylor series). *$f : A \to \mathbb{R}$, $c \in (a, b)$. f has n-th order derivative.*

Then, the n-th order Taylor series expansion is:

$$P_n(x) = f(c) + f'(c) \cdot (x - c) + \frac{f''(c)}{2!} \cdot (x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!} \cdot (x - c)^n$$

Problem: If $f(x)$ is a polynomial of degree n, then $f(x) = P_n(x)$.

Theorem 29 (Taylor with Largrange Remainder). *$f : A \to \mathbb{R}$, $c \in (a, b)$. f is $(n+1)$-th order differentiable on (a, b). For every $a < x < b$, there exists $\xi_{x,c}$ between x and c s.t.*

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_{x,c})}{(n+1)!}(x - c)^{n+1}$$

where $\xi_{x,c}$ is between x and c.

Proof. x, c are fixed. **Goal:** find $\xi_{x,c}$.

Define $g(t) = f(x) - f(t) - f'(t) \cdot (x - t) - \frac{f''(t)}{2!} \cdot (x - t)^2 - \cdots - \frac{f^{(n)}(t)}{n!} \cdot (x - t)^n$. Then, we have:

$$\begin{aligned} g(x) &= 0 \\ g(c) &= f(x) - P_n(x) \\ g'(t) &= -f'(t) - f''(t) \cdot (x - t) + f'(t) - \frac{f^{(3)}(t)}{2!} \cdot (x - t)^2 + f''(t) \cdot (x - t) - \dots \\ &= -\frac{f^{n+1}(t)}{n!} \cdot (x - t)^n \end{aligned}$$

Define $h(t) = g(t) - \left(\frac{x-t}{x-c}\right)^{n+1} \cdot g(c)$. Then, we have:

$$h(x) = h(c) = 0$$

By MVT, there exist some $\xi_{x,c}$ s.t. $h'(\xi_{x,c}) = 0$:

$$h'(\xi_{x,c}) = g'(\xi_{x,c}) + (n + 1)\frac{(x - \xi_{x,c})^n}{(x - c)^{n+1}} \cdot g(c) = 0$$

Thus, we have:

$$(n+1)\frac{(x-\xi_{x,c})^n}{(x-c)^{n+1}} \cdot g(c) = \frac{f^{(n+1)}(\xi_{x,c})}{n!} \cdot (x-\xi_{x,c})^n$$

Therefore, we have:

$$g(c) = \frac{f^{(n+1)}(\xi_{x,c})}{(n+1)!} \cdot (x-c)^{n+1}$$

□

Chapter 7

Sequences and series of functions

7.1 Uniform convergence

Definition 17 (Pointwise convergence)**.** *We say $f_n(x) \to f$ pointwisely if* $\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x$

$$\Leftrightarrow \forall x \in A, \forall \epsilon > 0 : \ \exists N_{\epsilon,x} > 0 \ s.t. \ |f_n(x) - f(x)| < \epsilon \quad \forall n > N_{\epsilon,x}$$

Definition 18 (Uniform convergence)**.** *We say $f_n(x) \to f(x)$ uniformly if* $\forall \epsilon > 0 \ \exists N_\epsilon > 0 \ s.t.$

$$|f_n(x) - f(x)| < \epsilon \quad \forall n > N_\epsilon, \forall x$$

Definition 19 (Cauchy condition for uniform convergence)**.** *We say $(f_n(x))$ is uniformly Cauchy if* $\forall \epsilon > 0$

$$\exists N_\epsilon > 0 \ s.t. \ |f_n(x) - f_m(x)| < \epsilon \quad \forall n, m > N_\epsilon, \forall x$$

Theorem 30. *Uniform convergence $\Leftrightarrow$ uniform Cauchy.*

Proof. $\Rightarrow$: $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \epsilon$.

$\Leftarrow$: Fix x. Note that $(f_n(x))$ is a Cauchy sequence, i.e. $(f_n(x))$ is a convergent sequence. Let $\lim_{n\to\infty} f_n(x) = f(x)$. Then, we have:

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \epsilon + 0 \quad \forall n > N, m \to \infty$$

□

Theorem 31. $\{f_n(x)\}, f_n(x) : A \to \mathbb{R}$ *each of f_n is bounded. If $f_n \to f$ uniformly, then f is also a bounded function.*

Proof. **Key:** show $|f(x)|$ is bounded, independent of x.

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < 1 + M \quad \forall x$$

Choose $\epsilon = 1$. Since $f_n \to f$ uniformly, there exists $N > 0$ s.t. $|f_n(x) - f(x)| < 1 \quad \forall n > N, \forall x$.

Choose $n = N + 1$. Note that f_{N+1} is bounded, i.e. $|f_{N+1}| < M \quad \forall x$. □

Theorem 32. *$\{f_n(x)\}, f_n(x) : A \to \mathbb{R}$ each of f_n is continuous. If $f_n \to f$ uniformly, then f is also a continuous function.*

Proof. **Key:** show $\lim_{x \to c} f(x) = f(c)$.

$$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| < \frac{2\epsilon}{3} + \frac{\epsilon}{3}$$

Since $f_n \to f$ uniformly, $\forall \epsilon > 0$ there exists $N > 0$ s.t. $|f_n(x) - f(x)| < \epsilon/3 \quad \forall n > N, \forall x$.

Choose $n = N + 1$: since f_{N+1} is a continuous function, $\forall \epsilon > 0 :\ \exists \delta_{N+1,\epsilon}$ s.t. $|f_{N+1}(x) - f_{N+1}(c)| < \epsilon/3$ if $|x - c| < \delta_{N+1,\epsilon}$. □

Theorem 33. *$\{f_n(x)\}, f_n(x) : A \to \mathbb{R}$ each of f_n is differentiable. Assume $f_n \to f$ pointwisely, $f_n' \to g$ uniformly. Then, f is also differentiable and $f' = g$.*

Proof. **Key:** show $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = g(c)$.

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + |f_n'(c) - g(c)|$$

(1) Note that we can choose arbitrary $m \to \infty$ so that first term goes to 0:

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_m(x) - f_m(c)}{x - c} \right| + \left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right|$$

By MVT for the second term, there is some ξ between x and c s.t. $(f_m - f_n)'(\xi) = \frac{(f_m-f_n)(x)-(f_m-f_n)(c)}{x-c}$.

Since $f_n' \to g$ uniformly, $(f_n'(x))$ is uniformly Cauchy: $\forall \epsilon > 0 : \exists N$ s.t.

$$\forall n, m > N_2 : \quad |f_n'(x) - f_m'(x)| < \epsilon/3 \quad \forall x$$

(2) Choose sufficiently large $n > \max(N_1, N_2)$ Since f_n is differentiable, $\exists \delta > 0$ s.t. if $|x - c| < \delta$ then $\left| \frac{f_n(x)-f_n(c)}{x-c} - f_n'(c) \right| < \epsilon/3$.

(3) $f_n' \to g$ uniformly, then $\forall \epsilon > 0 : \exists N_2 > 0$ s.t. $|f_n'(x) - g(x)| < \epsilon/3 \quad \forall n > N_2, \forall x$.

□

7.2 Series of functions

Definition 20. $\{f_n(x)\}, f_n(x) : A \to \mathbb{R}$.

We say that $\sum_{n=1}^{\infty} f_n(x)$ *converges pointwisely if* $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)$ *converges pointwisely.*

Theorem 34. $\{f_n(x)\}, f_n(x) : A \to \mathbb{R}$.

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges uniformly} \quad \Leftrightarrow \quad \forall \epsilon > 0 : \ \exists N > 0 \text{ s.t. } \left| \sum_{k=m+1}^{n} f_k(x) \right| < \epsilon \quad \forall n > m > N, \forall x$$

Proof. $\Rightarrow$: Note that $(S_n(x))$ ia a uniform Cauchy sequence. Then, (S_n) converges.

Fix x. Apply Cauchy condition for series: $S_n(x) - S_m(x)$. □

Theorem 35 (Comparison test)**.** $\{f_n(x)\}, f_n(x) : A \to \mathbb{R}$.

If we can find $\{M_n\}$ *s.t.* $M_n \geq 0$, $\sum_{n=1}^{\infty} M_n < \infty$ *and* $|f_n(x)| < M_n \quad \forall n, \forall x$, *then* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly.*

Proof. Because $\sum M_n < \infty$ converges, $\forall \epsilon > 0 : \exists N > 0$ s.t. we have

$$\left| \sum_{k=m+1}^{n} f_k(x) \right| \leq \sum_{k=m+1} |f_k(x)| < \sum_{k=m+1}^{n} M_k < \epsilon \quad \forall n > m > N$$

□

7.3 Power series

$$\sum_{n=1}^{\infty} a_n(x-c)^n$$

where $c \in \mathbb{R}$, $\{a_n\} \subseteq \mathbb{R}$.

There always exists radius of convergence R, radius of uniform convergence $0 \le \rho < R$.

Proposition 8. *If $\sum_{n=1}^{\infty} a_n x_0^n$ converges for some $x_n \in \mathbb{R}$, where $|x_0| > 0$, then $\sum_{n=1}^{\infty} a_n x^n$ converges for any $|x| < |x_0|$.*

Proof. Since $\sum_{n=1}^{\infty} a_n x_0^n$ converges, $\lim_{n\to\infty} a_n x_0^n = 0$, i.e. a sequence $(a_n x_0^n)$ is bounded.

$\exists M > 0$ s.t. $|a_n x_0^n| < M \quad \forall n$.

$|a_n x^n| = |a_n x_0^n \cdot (\frac{x}{x_0})^n| < M \cdot \left|\frac{x}{x_0}\right|^n \to 0$ as $n \to \infty$. □

Theorem 36. *Define $f_n(x) = a_n(x-c)^n$. Then, we can find $R \ge 0$ s.t.*

(1) if $|x| < R$, $\sum_{n=1}^{\infty} |f_n(x)|$ converges, if $|x| > R$, $\sum_{n=1}^{\infty} f_n(x)$ diverges.

(2) if $|x| \le \rho$, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Proof. (1) Define $R = \sup\{|x| : \sum_{n=1}^{\infty} a_n x^n\}$.

Let $0 < R < \infty$. By definition of sup, $\exists x_0$ s.t. $|x| < |x_0| < R$ and $\sum_{n=1}^{\infty} a_n x_0^n$ converges. When $|x| > R$, because R is the upper bound, we must have $|x| \notin \{|x| : \sum_{n=1}^{\infty} a_n x^n \text{ converges}\}$.

Let $R = 0$. $\sum_{n=1}^{\infty} a_n x_0^n$ diverge always.

Let $R = \infty$. $\exists x_0$ s.t. $|x_0| > |x|$ always, i.e. always converges.

(2) Given $\rho < R$, since R is sup, $\exists x_0$ s.t. $\rho < |x_0| < R$ and $\sum a_n x_0^n$ converges. Then, $\lim_{n\to\infty} a_n x_0^n = 0$ and $\exists M > 0$ s.t. $|a_n x_0^n| < M \quad \forall n$.

$|a_n x^n| = |a_n x_0^n \cdot (\frac{x}{x_0})^n| \le M \cdot \left|\frac{x}{x_0}\right|^n = M \cdot \left|\frac{\rho}{x_0}\right|^n \to 0$ as $n \to \infty, \forall |x| < \rho$, i.e. independent of x.

□

Theorem 37. *(1) $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if limit exists.*

(2) $R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$.

Proof. (1) According to ratio test, converges if $\lim \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim \left|\frac{a_{n+1}}{a_n}\right| \cdot |x| < 1$.

(2) According to root test, converges if $\limsup \sqrt[n]{a_n|x|^n} = \limsup \sqrt[n]{a_n} \cdot |x| < 1$

□

Proposition 9. $\sum_{n=1}^{\infty} a_n x^n$ *has the same convergent radius R as (its derivative) $a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n$.*

Proof. Note that limsup always exists.

$$R = \frac{1}{\limsup \sqrt[n]{a_n}} = \frac{1}{\limsup \sqrt[n]{na_n}} = \frac{1}{\limsup \sqrt[n+1]{(n+1)a_n}}$$

□

Theorem 38. *$f(x)$ is infinite-differentiable on $(-R, R)$.*

Proposition 10. *$R,\ S > 0$, functions $f(x) = \sum_{n=1}^{\infty} a_n x^n$ in $|x| < R$,*

Chapter 8

Integrable functions

8.1 Supremum and infimum of functions

Proposition 11. *If $f, g : A \to \mathbb{R}$ are bounded functions, then:*

$$\left|\sup_A f - \sup_A g\right| \le \sup_A |f-g|, \quad \left|\inf_A f - \inf_A g\right| \le \sup_A |f-g| \tag{8.1}$$

Proof. $\sup f \le \sup(f-g) + \sup g \le \sup |f-g| + \sup g$. Use $\sup(-f) = -\inf f$. □

8.2 Riemann integrable

Definition 21 (Upper and lower Riemann sum)**.**

$$\begin{aligned} U(f;P) &= \sum_{k=1}^{n} \sup_{[x_{k-1},x_k]} f \cdot (x_k - x_{k-1}) \quad \textit{and} \quad U(f) = \inf_P U(f;P) \\ L(f;P) &= \sum_{k=1}^{n} \inf_{[x_{k-1},x_k]} f \cdot (x_k - x_{k-1}) \quad \textit{and} \quad L(f) = \sup_P L(f;P) \end{aligned} \tag{8.2}$$

Definition 22. *A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if it is bounded and $U(f) = L(f)$.*

Definition 23. *A partition $Q = \{J_1, \ldots, J_m\}$ is a refinement of a partition $P = \{I_1, \ldots, I_n\}$ if every interval I_k in P is an almost disjoint union of one or more intervals J_l in Q ($m \ge n$).*

Theorem 39. *Let $f : [a,b] \to \mathbb{R}$ be bounded, P be a partition on $[a,b]$, Q be a refinement of P. Then*

$$U(f;Q) \le U(f;P), \quad L(f;P) \le L(f;Q) \tag{8.3}$$

Proposition 12. *If $f : [a,b] \to \mathbb{R}$ is bounded and P, Q are partitions of $[a,b]$, then $L(f;P) \le U(f,Q)$.*

Proof. Let R be a common refinement of P and Q. Then: $L(f;P) \le L(f,R) \le U(f,R) \le U(f,Q)$. □

Proposition 13. *If $f : [a, b] \to \mathbb{R}$ is bounded, then $L(f) \leq U(f)$.*

Proof. Consider $A = \{L(f; P) : P \in \Pi\},\ B = \{U(f; P) : P \in \Pi\}$.

From previous proposition, $L \leq U \quad \forall L \in A, U \in B$. Thus, $L(f) = \sup A \leq \inf B = U(f)$. □

8.3 Cauchy condition for integrability

Theorem 40. *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff for every $\epsilon > 0$ there exists a parition P of $[a, b]$ (may depend on ϵ) s.t.*

$$U(f; P) - L(f; P) < \epsilon \tag{8.4}$$

Proof. $\Leftarrow$: Given that Cauchy condition holds for $\epsilon > 0$ and a partition P.

Note that $U(f) \leq U(f; P), L(f) \geq L(f; P)$. Then, we have:

$$0 \leq U(f) - L(f) \leq U(f; P) - L(f; P) < \epsilon$$

so $U(f) = L(f)$, and f is Riemann integrable by definition.

$\Rightarrow$: Given that f is integrable. Let $\epsilon > 0$. Then, there exist partitions Q, R s.t.

$$U(f; Q) < U(f) + \frac{\epsilon}{2}, \quad L(f; R) > L(f) - \frac{\epsilon}{2}$$

Let P be a common refinement of Q and R. Then:

$$U(f; P) - L(f; P) \leq U(f; Q) - L(f; R) \leq U(f) - L(f) + \epsilon$$

where $U(f) = L(f)$ by definition. □

8.4 Continuous, monotonic functions

Theorem 41. *A continuous function $f : [a, b] \to \mathbb{R}$ on a compact interval is Riemann integrable.*

Proof. A contitnuous function on a compact set is bounded and uniformly continuous.

Thus, $\forall > 0 : \exists \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \forall x, y \in [a, b]$ satisfying $|x - y| < \delta$.

Choose a partition $P = \{I_1, \ldots, I_n\}$ s.t. $|I_i| < \delta \quad \forall i$.

A function attains its max value M_k at x_k and min value m_k at y_k in compact interval I_k. Since $|x_k - y_k| < \delta$, we have $M_k - m_k < \frac{\epsilon}{b-a}$:

$$U(f; P) - L(f; P) = \sum_{k=1}^{n} M_k |I_k| - \sum_{k=1}^{n} m_k |I_k| = \sum_{k=1}^{n} (M_k - m_k)|I_k| < \frac{\epsilon}{b-a} \sum_{k=1}^{n} |I_k| < \epsilon$$

□

Theorem 42. *A monotonic function $f : [a, b] \to \mathbb{R}$ on a compact interval is Riemann integrable.*

Proof. Let f be monotonic increasing: $f(x) \leq f(y) \quad \forall x \leq y$.

Choose a partition $P_n = \{I_1, \ldots, I_n\}$ of n subintervals $I_k = [x_{k_1}, x_k]$ of $[a, b]$, each of length $\frac{b-a}{n}$ with endpoints

$$x_k = a + (b-a)\frac{k}{n} \quad k = 0, 1, 2, \ldots, n$$

Since f is increasing, $M_k = \sup_{I_k} f = f(x_k)$, $m_k = \inf_{I_k} f = f(x_{k-1})$.

$$U(f; P_n) - L(U; P_n) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) = \frac{b-a}{n}\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})] = \frac{b-a}{n}[f(b) - f(a)]$$

which goes to 0 as $n \to \infty$. □

8.5 Properties

Theorem 43. *Given $f,\ g : [a, b] \to \mathbb{R}$ are Riemann integrable.*

(1) fg is Riemann integrable

(2) $\frac{f}{g}$ is Riemann integrable if $\frac{1}{g}$ is bounded

Proof. (1) Since $f,\ g$ are Riemann integrable, $\exists M > 0$ s.t. $|f|,\ |g| \leq M$.

Pick $\forall x, y \in A = [t_{k-1}, t_k]$:

$$\begin{aligned} f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y) &\leq M(|f(x) - f(y)| + |g(x) - g(y)|) \\ &\leq M((\sup f - \inf f) + (\sup g - \inf g)) \end{aligned}$$

Note that $|f(x) - f(y)| \leq \sup f - \inf f \quad \forall x, y$.

Then, we have:

$$U(fg, P) - L(fg, P) \leq M \cdot [U(f, P) - L(f, P) + U(g, P) - L(g, P)] \quad \forall P$$

Since $f,\ g$ are Riemann integrable, $\forall \epsilon > 0 :\ \exists R$ s.t.

$$U(f, R) - L(f, R) \leq \frac{\epsilon}{2M}$$
$$U(g, R) - L(g, R) \leq \frac{\epsilon}{2M}$$

□

8.5.1 Linearity, monotonicity, additivity

Theorem 44 (Linearity). *$f : [a, b] \to \mathbb{R}$ is integrable and $c \in \mathbb{R}$, then cf is integrable and*

$$\int_a^b cf = c\int_a^b f$$

Proof. Observe $c \geq 0$. Observe $-f$, where $\sup(-f) = -\inf(f)$. Then, observe $c < 0$ where $c = -|c|$. □

Theorem 45 (Monotonicity). *$f,\ g : [a, b] \to \mathbb{R}$ are Riemann integrable and $f \leq g \quad x \in [a, b]$. Then*

$$\int_a^b f \leq \int_a^b g$$

Proof. Define $h(x) = g(x) - f(x) \geq 0$, which is also Riemann integrable by linearity.

Then, we have $\inf_{[t_{k-1},t_k]} h \geq 0$. So, $L(f, P) \geq 0 \quad \forall P$.

By definition, $\int_a^b h = \sup_P L(h, P) \geq 0$, so $\int_a^b g - \int_a^b f = \int_a^b h \geq 0$. □

Theorem 46 (Monotonicity). *If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then $|f|$ is also Riemann integrable and*

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Proof.

$$|f(x)| - |f(y)| \leq \sup_A f - \inf_A f \quad \forall x, y \in A$$

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) \quad \forall P$$

By definition, we have:

$$-\int_a^b |f|dx \leq \int_a^b f dx \leq \int_a^b |f|dx$$

i.e. $-b \leq a \leq b \Rightarrow |a| \leq b$. □

Theorem 47 (Additivity). *Choose $c \in (a, b)$.*

f is Riemann integrable on $[a, b]$ $\Leftrightarrow$ f is Riemann integrable on $[a, c]$ and $[c, b]$, where

$$\int_a^c f dx + \int_c^b f dx = \int_a^b f dx$$

Proof. $\Rightarrow$: Given f is integrable on $[a, b]$. Thus, $\forall \epsilon > 0 : \exists P$ s.t. $U(f; P) - L(f; P) < \epsilon$.

Let $P' = P \cup \{c\}$, and split $P' = Q \cup R$ into partitions Q of $[a, c]$ and R of $[c, b]$.

$$U(f; P') = U(f; Q) + U(f; R) \text{ and } L(f; P') = L(f; Q) + L(f; R)$$

$$U(f; Q) - L(f; Q) = U(f; P') - L(f; P') - [U(f; R) - L(f; R)] < U(f; P) - L(f; P) < \epsilon$$

$\Leftarrow$: Given f is integrable on $[a, c]$ and $[c, b]$. Let $P = Q \cup R$.

$$U(f; Q) - L(f; Q) < \frac{\epsilon}{2} \text{ and } U(f; R) - L(f; R) < \frac{\epsilon}{2}$$
$$U(f; P) - L(f; P) = [U(f; Q) + U(f; R)] - [L(f; Q) + L(f; R)] < \epsilon$$

$\square$

8.6 Fundamental theorem of calculus

Theorem 48 (Fundamental theorem of calculus)**.**

Version 1. $F(x) : [a, b] \to \mathbb{R}$ *differentiable,* $f(x) = F'(x)$ *integrable, then* $\int_a^b f(x)dx = F(b) - F(a)$.

Version 2. $f(x) : [a, b] \to \mathbb{R}$ *integrable, define* $F(x) = \int_a^x f(t)dt$*. Then:*

1. $F(x)$ *is a continuous function on* $[a, b]$
2. *if* $f(x)$ *is continuous at* $c \in [a, b]$*, then* $F'(c) = f(c)$

Proof. (Version 1.) Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$. Then, $F(b) - F(a) = \sum_{k=1}^n [F(x_k) - F(x_{k-1})]$.

By MVT, $\exists c \in (x_{k-1}, x_k)$ s.t. $F(x_k) - F(x_{k-1}) = f(c)(x_k - x_{k-1})$:

$$L(f; P) = m_k(x_k - x_{k-1}) \leq F(x_k) - F(x_{k-1}) \leq M_k(x_k - x_{k-1}) = U(f; P)$$

where f is Riemann integrable, i.e. $U(f) = L(f)$, and thus $F(x_k) - F(x_{k-1}) = \int_{x_{k-1}}^{x_k} f(x)dx$. □

Proof. (Version 2.)

(1) **Key:** derive Lipschitz! Since f is Riemann integrable, it is bounded. Let $|f(t)| \leq M \quad \forall t$.

$$|F(y) - F(x)| = \left|\int_x^y f(t)dt\right| \leq \int_x^y |f(t)|dt \leq \int_y^x M dt = M(x - y) \quad \forall x < y \tag{8.5}$$

so F is continuous.

(2) Given $c \in [a, b]$. Define

$$\frac{F(c + h) - F(c)}{h} = \frac{1}{h}\int_c^{c+h} f(t)dt$$

Then, we have:

$$\frac{F(c + h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} f(t)dt - \frac{1}{h}\int_c^{c+h} f(c)dt = \frac{1}{h}\int_c^{c+h} [f(t) - f(c)]dt \tag{8.6}$$

Since f is continuous at c, $\forall \epsilon > 0 : \exists \delta > 0$ s.t. $|f(t) - f(c)| < \epsilon \quad \forall |t - c| < \delta$.

Choose $h < \delta$:

$$\left|\frac{1}{h}\int_c^{c+h} [f(t) - f(c)]dt\right| \leq \frac{1}{h}\int_c^{c+h} |f(t) - f(c)|dt \leq \frac{1}{h}\int_c^{c+h} \epsilon dt = \frac{1}{h}\epsilon h = \epsilon \tag{8.7}$$

□

and $F'(c) = \lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(t)dt = f(c)$.

Theorem 49 (Integration by parts)**.** *$f,\ g:[a,b]\to\mathbb{R}$ differentiable, $f',\ g'$ integrable. Then*

$$\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx$$

Proof.

$$\int_a^b (fg' + f'g)dx = \int_a^b (fg)'dx = f(b)g(b) - f(a)g(a) \tag{8.8}$$

□

Theorem 50 (Change of variable)**.** *f continuous, g differentiable. Then*

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))\cdot g'(x)dx = \int_a^b F'(g(x))dx = F(g(b)) - F(g(a))$$

8.7 Last class: integration of sequences of functions

Theorem 51. *$f_n:[a,b]\to\mathbb{R}$ integrable for all n, $f_n\to f$ uniformly on $[a,b]$ as $n\to\infty$. Then $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and*

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$$

Proof. Since $f_n\to f$ uniformly, $\exists N>0$ s.t.

$$f_n(x) - \frac{\epsilon}{b-a} < f(x) < f_n(x) + \frac{\epsilon}{b-a} \quad \forall x\in[a,b]$$

Note that $L(f_n - \frac{\epsilon}{b-a}) \le L(f)$, $U(f_n + \frac{\epsilon}{b-a}) \ge U(f)$.

Since f_n is integrable and upper Riemann sums are greater than lower Riemann sums:

$$\int_a^b f_n - \epsilon \le L(f) \le U(f) \le \int_a^b f_n + \epsilon \quad \forall n > N$$

so $0\le U(f) - L(f) \le 2\epsilon$, thus $U(f) = L(f)$ and f is Riemann integrable by definition.

Note that $\left|\int_a^b f_n - \int_a^b f\right| \le \epsilon \quad \forall n > N$, so $\int_a^b f_n \to \int_a^b f$ as $n\to\infty$. □

Chapter 9

Final

Problem: Assume that $f(x)$ is the first-order differentiable ($f'(x)$ exists) in $[-1, 1]$ and $\sum_{n=1}^{\infty} f(\frac{1}{n})$ absolutely converges. Prove that $f'(0) = 0$.

Hint: notice that $\sum \frac{c}{n}$ is convergent and use proof by contradiction.

Solution:

Proof. Since $\sum f(\frac{1}{n})$ converges, $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 = f(0)$, where f is continuous at 0.

(Proof by contradiction.) Assume $f'(0) = c \neq 0$.

Then, we have:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = c$$

$$\forall \epsilon > 0 : \exists \delta > 0 \text{ s.t. } \left| \frac{f(x)}{x} - c \right| < \epsilon = \frac{c}{2} \quad \forall |x - 0| < \delta$$

$$\left| \frac{f(x)}{x} \right| > \frac{|c|}{2} \quad \forall |x| < \delta$$

Replace $x = \frac{1}{n}$, i.e. $|x| < \delta$ is the same as $n > \frac{1}{\delta}$. Then, we have:

$$\left| f\left(\frac{1}{n}\right) \right| > \frac{|c|}{2n} \quad \forall n > \frac{1}{\delta}$$

Note that $\frac{|c|}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

By comparison test, $\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$ diverges, too. Contradiction.

□

Problem: Let f be a bounded function on $[0,1]$. Given any partition $P = \{0 = p_0 < p_1 < \cdots < p_m = 1\}$ on $[0,1]$, we define

$$\text{len}(P) = \max_{o \leq k \leq m-1} p_{k+1} - p_k$$

We also define

$$U_n = \inf_{P,\ \text{len}(P) \geq \frac{1}{n}} U(f;P), \quad L_n = \sup_{P,\ \text{len}(P) \geq \frac{1}{n}} L(f;P)$$

Prove:

(a) $\lim_{n\to\infty} U_n$ and $\lim_{n\to\infty} L_n$ exist.

(b) f is Riemann integrable on [0, 1] iff $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.

Solution:

(a) *Proof.* Note that U_n is defined over smaller and smaller partitions, thus decreasing. Since U_n is bounded below by $\inf_{[0,1]} f \cdot (1-0)$, $\lim_{n\to\infty} U_n$ exists.

Similarly, L_n is increasing and bounded above by $\sup_{[0,1]} f$, and thus $\lim_{n\to\infty} L_n$ exists. □

(b) *Proof.* $\Rightarrow$: Given f is integrable. Then, Cauchy condition for integrability holds:

$$\forall \epsilon > 0 : \exists P \text{ s.t. } U(f;P) - L(f;P) < \epsilon$$

We increase max subinterval of P further s.t. $\text{len}(P) \geq \frac{1}{n}$: $U_n \leq U(f;P)$, $L_n \geq L(f;P)$.

$$U_n - L_n \leq U(f;P) - L(f;P) < \epsilon$$

and choosing $n \to \infty$ yields $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.

$\Leftarrow$: Given $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$. Then, $\lim_{n\to\infty} U_n - L_n = 0$.

By def of limit, $\forall \epsilon > 0 : \exists N > 0$ s.t. $|U_n - L_n - 0| < \epsilon \quad \forall n > N$.

By def of limsup/liminf, $\exists P, Q$ s.t. $\text{len}(P), \text{len}(Q) \geq \frac{1}{n}$: $U(f;P) \leq U_n + \frac{\epsilon}{2}$, $L(f;Q) \geq L_n - \frac{\epsilon}{2}$.

Let $R = P \cup Q$ be a common refinement:

$$U(f;R) - L(f;R) \leq U(f;P) - L(f;Q) \leq U_n - L_n + \epsilon$$

□

Problem: Let $f(x)$ be first-order differentiable on $[a,b]$, $f(a) \neq 0$, $f(b) \neq 0$.
Define a sequence $\{x_n\}$ s.t. $f(x_n) = 0$ for all n. Let $\liminf_{n\to\infty} x_n = c$.

(a) Prove $c \in (a,b)$. *Note*: liminf is not necessarily part of the sequence!

(b) Prove $f(c) = f'(c) = 0$.

Solution:

(a) *Proof.* Since f is continuous at a, $f(x) \neq 0 \quad \forall x \in [a-\delta, a+\delta]$. Similarly for b.
So, $\{x_n\} \in [a+\delta, b-\delta_1] \subset (a,b)$ and liminf of $\{x_n\}$ must be inside (a,b). □

(b) *Proof.* Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ s.t. $\lim_{k\to\infty} x_{n_k} = c$. Then

$$f(c) = f(\lim_{k\to\infty} x_{n_k}) = 0$$

and $f'(c) = \lim_{k\to\infty} \frac{f(x_{n_k})-f(c)}{x-c} = 0$ (since it is well-defined). □

Problem: Let $f:(a,b) \to \mathbb{R}$ be differentiable, $|f'(x)| \leq M$ for some $M > 0$. Prove: $\lim_{x\to b^-} f(x)$ exists.

Solution:

Proof. Consider $a < x < y < b$: f is continuous on $[x,y]$ differentiable on (x,y).
By MVT, $\exists c \in (x,y)$ s.t. $|f'(c)| = \left|\frac{f(y)-f(x)}{y-x}\right| \leq M$. Then, $|f(y)-f(x)| \leq M|y-x| \quad \forall x,y \in (a,b)$.
For any $\epsilon > 0$, choose $\delta = \frac{M}{\epsilon}$.
Then, f is uniformly continuous: $|f(y)-f(x)| \leq M|y-x| = \epsilon \quad \forall |x-y| < \delta$.
Choose a sequence $(x_n) \subset (a,b)$ s.t. $x_n \to b^-$ (consider $x_n = b - 1/n$).
Then, since (x_n) converges, (x_n) is a Cauchy sequence: $\forall \epsilon > 0 : \exists N > 0$ s.t. $|x_n - x_m| < \epsilon \quad \forall n,m > N$.
Thus, the Cauchy condition for uniform convergence of f holds:

$$|f(x_n) - f(x_m)| \leq \epsilon \quad \forall n,m > N$$

since $|x_n - x_m| < \delta$. Thus, the sequence $(f(x_n))$ converges and $\lim_{x\to b^-} f(x_n)$ exists.

Note that since f is uniformly continuous, for different sequences $x_n \to b^-$ and $y_n \to b^-$, we have $\forall \epsilon > 0 : \exists \delta > 0$ s.t. $|f(x_n) - f(y_n)| < \epsilon \quad \forall |x_n - y_n| < \delta$, and so

$$\lim_{x_n \to b^-} f(x_n) = \lim_{y_n \to b^-} f(y_n)$$

□