Contents

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Chapter 1

Vector spaces

1.1 Formal systems

By definition, $\mathbb{R}$ and $\mathbb{C}$ are fields.

Proposition. Properties of complex arithmetic:

- commutativity
- associativity
- identities
- additive inverse
- multiplcative inverse
- distributive property

Definition. A *field* is a set with at least two distinct elements 0 and 1, and two operations of addition and multplication, satisfying all properties in Proposition 1.

Definition. Let G be a set. A binary operation is a map of sets: $* : G \times G \to G$.

Addition $+$ and multiplication $\times$ are binary operations.

Theorem (Remainder Theorem)**.** Given $a, b \in \mathbb{Z}$, if $b > 0$, then $\exists! q, r \in \mathbb{Z}$ such that $a = bq + r$ with $0 \leq r < b$.

Lemma (Euclid's Lemma)**.** Let p be a prime number and $a, b \in \mathbb{Z}$. Then

$$p|ab \implies p|a \text{ or } b$$

Theorem (Fundamental Theorem of Arithmetic)**.** Every positive integer $a > 1$ can be written as a product of primes

$$a = p_1 p_2 \ldots$$

uniquely up to ordering.

Definition. A group is a set G with a binary operation $*$ such that

1. $(a * b) * c = a * (b * c) \quad \forall a, b, c \in G$ (associativity)
2. $\exists e \in G$ such that $a * e = e * a = a \quad \forall a \in G$ (identity)
3. given $a \in G : \exists b \in G$ such that $a * b = b * a = e$ (inverse)

A set with a binary operation is called a *monoid* if only the first two properties hold.

Definition. A group $(G, *)$ is called Abelian if it also satisfies

$$a * b = b * a \quad a, b \in G$$

(commutative property).

Definition. A ring is a set R with two binary operations $+$ (addition) and $\times$ (multiplication) such that

1. R is an Abelian group under addition
2. R is a monoid under multiplcation (inverses do not necessarily exist)
3. $+$ and $\times$ are related by the distributive law

$$(x + y) \times z = x \times z + y \times zx \times y + x \times z \quad \forall x, y, z \in R$$

Isomorphism theorem

Definition. Let R be a ring. An ideal $I \subset R$ is a subset which is a subgroup under addition and is closed under both left and right multiplicaiton by all of R.

$$\forall x \in I : xr, rx \in I \quad \forall r \in R$$

Proposition. $R \backslash I$ is a ring under the natural operations. We call it the quotient ring.

Definition. A field is a commutative division ring.

Polynomial ring
Galois theory

$(\mathbb{Z}, +)$: an Abelian group, $(\mathbb{Z}, +, \times)$: a ring, $(\mathbb{Q}, +, \times)$: field.
$(\mathbb{R}^n, +, \text{scalar multiplication})$: vector spaces over $\mathbb{R}$.

1.2 Definition of vector space

The elements of the field F are called *scalars*, and the elements of a vector space V are called *vectors*.

Proposition. $O_F \in F,\ 0_V \in V$.

Example. Let $\mathcal{F} = \{\text{all functions } f : \mathbb{R} \to \mathbb{R}\}$. $(\mathcal{F}, +, \cdot)$ is a vector space over $\mathbb{R}$.

More generally, define some set S and some field F. Let $\mathcal{F}^S = \{\text{all functions } f : S \to F\}$. Then, $(\mathcal{F}^S, +, \cdot)$ is a vector space over F.

Definition. A vector space V over a field F is a set equipped with two operations $+$ (addition) and $\cdot$ (scalar multiplication) s.t. $\forall x, y \in V : \exists x + y \in V$ and $\forall a \in F, \forall x \in V : \exists ax \in V$ (i.e. closed under addition and scalar multiplication) where following conditions hold:

1. $\forall x, y \in V,\ x + y = y + x$ (commutativity)
2. $\forall x, y, z \in V,\ (x + y) + z = x + (y + z)$ (associativity)
3. $\exists 0 \in V$ such that $x + 0 = x \quad \forall x \in V$ (zero element)
4. $\forall x \in V : \exists y \in V$ such that $x + y = 0$ (additive inverse)
5. $\forall x \in V,\ 1 \cdot x = x$ (identity)
6. $\forall a, b \in F, \forall x \in V,\ (a \cdot b) \cdot x = a \cdot (b \cdot x)$
7. $\forall a \in F, \forall x, y \in V,\ a \cdot (x + y) = a \cdot x + a \cdot y$
8. $\forall a, b \in F, \forall x \in V,\ (a + b) \cdot x = a \cdot x + b \cdot x$

Proposition. Both $+$ (addition) and $\cdot$ (scalar multiplication) satisfy axioms:

(1) commutativity $\forall u \in V, \forall v \in V : u + v = v + u$

(2) associativity

(3) $\exists 0_V \in V : v + 0_V = v \quad \forall v \in V$

(4) $\forall v \in V : \exists w \in V$ s.t. $v + w = 0_V$

Theorem (Uniqueness of $O_V \in V$)**.** Suppose $0 \in V$ and $0' \in V$ both satisfy (3). Then $0 = 0'$.

Proof.

$$0' = 0' + 0 = 0 + 0' = 0$$

□

Theorem (Uniqueness of additive inverse)**.** Let $v \in V$; suppose $v + w = 0$ and $v + w' = 0$. Then $w = w'$.

Proof.

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w' + 0 = w'$$

□

(Set theory) Let S be a set. Then we say that set T is a subset of S if T consists of some of the elements of S (and nothing else):

$$T \subset S \iff \forall a \in T : a \in S$$

1.3 Subspaces

Suppose $(V, +, \cdot)$: a vector space over some field F.

Theorem. A subspace of a vector space is also a vector space.

Theorem. $U \subset V$ is a subspace iff:

(a) $0_V \in U \iff U$ is nonempty

(b) U is closed under $+$

(c) U is closed under $\cdot$

Definition (Direct sum)**.** Given $V_1, \ldots, V_m$: subspaces of V. The sum $V_1 + \cdots + V_m$ is called a direct sum $V_1 \oplus \cdots \oplus V_m$ if each element of $V_1 + \cdots + V_m$ can be written in only one way as a sum $v_1 + \cdots + v_m$ where each $v_k \in V_k$.

Theorem. Given U, W: subspaces of V. Then $U + W$ is a direct sum $\iff U \cap W = \{0\}$.

Proof.
"$\Rightarrow$": Given $U \oplus W$.

Consider any $v \in U \cap W$, then $0 = v + (-v)$ where $v \in U, -v \in W$. Note that 0 has a unique representation as a sum of vector in U and a vector in W. Then, we have $v = 0$. Thus $U \cap W = \{0\}$. ■

"$\Leftarrow$": Given $U \cap W = \{0\}$.

Consider any $u \in U, w \in W$ s.t. $0 = u + w$ i.e. $u = -w \in W$ or equivalently $u \in U \cap W = \{0\}$. Then, we can represent any $u + w$ only as a direct sum.

□

Chapter 2

Finite-dimensional vector spaces

Definition. V, vector space over F, is called finite-dimensional if $V = \operatorname{span}(v_1, \ldots, v_n)$.

Example. Below subspace is not finite-dimensional (it is ∞-dimensional).

$$\mathcal{P}(\mathbb{R}) = \operatorname{span}(p_1(z), \ldots, p_n(z))$$

where $m_1, \ldots, m_n$: degrees.

Any linear combination will have degree $\leq \max(m_i)$.

However, $\mathcal{P}(\mathbb{R}) = \operatorname{span}(1, z, z^2, \ldots, z^m, \ldots)$, basis of $\mathcal{P}(\mathbb{R})$, is ∞-dimensional.

Choosing $m \in \{0, 1, 2, \ldots\}$:

$$\mathcal{P}_m(\mathbb{R}) = \left\{p(z) = b_0 + b_1 z + \cdots + b_m z^m \;\middle|\; b_i \in \mathbb{R}\right\} = \operatorname{span}(1, z, z^2, \ldots, z^m)$$

where $\mathcal{P}_m(\mathbb{R})$ is finite-dimensional.

2.1 Span and linear independence

List of vectors in V is an ordered collection $(v_1, v_2, \ldots, v_n)$ of vectors in V. Stipulate: $\operatorname{span}() = \{0_V\} \subset V$.

Definition. Let V be a vector space over F. Let $(v_1, \ldots, v_n)$ be a list of vectors in V.

(1) A linear combination is

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n = \sum_{i=1}^{n} a_i \cdot v_i$$

where each $a_i \in F$.

(2) A $\operatorname{span}(v_1, \ldots, v_n)$ is the set of all linear combinations of $(v_1, \ldots, v_n)$

$$\left\{\sum_{i=1}^{n} a_i \cdot v_i | a_i \in F\right\}$$

Lemma (Criterion for a subspace)**.** A subset $W \subset V$ is a *subspace* iff:

(a) W is closed under addition

(b) W is closed under scalar multiplication

(c) $0_V \in W$

Theorem. Let V be a vector space over F. If U is any subspace of V that contains all vectors $v_1, \ldots, v_n \in V$, then

$$\mathrm{span}(v_1, \ldots, v_n) \subset U$$

(1) $\mathrm{span}(v_1, \ldots, v_n)$ is a subspace of V.

(2) This is the *smallest* subspace $U \in V$ such that $v_i \in U \quad \forall i = 1, \ldots, n$.

Proof.
(1) Use the criterion for subspace:

(a) $\mathrm{span}(v_1, \ldots, v_n)$ is closed under addition

(b) $\mathrm{span}(v_1, \ldots, v_n)$ is closed under scalar multiplication

(c) $0_V \in \mathrm{span}(v_1, \ldots, v_n)$

(2) Let $U \subset V$ be some subspace s.t. $v_i \in U$ $\forall i = 1, \ldots, n$.
Note that:

$$\mathrm{span}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} a_i \cdot v_i \;\middle|\; a_i \in F, v_i \in U \right\}$$

Thus, $\mathrm{span}(v_1, \ldots, v_n) \subset U$, since U is closed under addition and scalar multiplication. $\square$

Definition. $(v_1, \ldots, v_n) \subset V$ is *linearly independent* if

$$a_1 \cdot v_1 + \cdots + a_n \cdot v_n = 0_V$$

has a unique solution $a_i = 0 \quad \forall i = 1, \ldots, n$ (no nontrivial solution). Otherwise, it is linearly dependent.

Theorem. $(v_1, \ldots, v_n) \in V$: linearly dependent. Then $\exists k \in \{1, \ldots, n\}$ s.t. $v_k \in \mathrm{span}(v_1, \ldots, v_{k-1})$.

Theorem. V: vector space over F. If $(u_1, \ldots, u_m)$: linearly independent list in V, $(w_1, \ldots, w_n)$: spanning list in V, then $m \leq n$ always.

2.2 Bases

Definition. A basis of V is a list of vectors in V that is linearly independent and spans V.

Theorem.

(1) Every spanning list in a vector space can be reduced to a basis of the vector space.

(2) Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Theorem. Any two bases of a finite-dimensional vector space have the same length.

Proof. From replacement theorem. □

Remark. Every finite-dimensional vector space V has a basis.

Lemma (Criterion for a basis)**.** A list of vectors $(v_1, \dots, v_n)$ in V is a basis iff every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n$$

where $a_1, \dots, a_n \in F$.

Proof. "$\Rightarrow$": Given $(v_1, \dots, v_n)$ is a basis of V. Consider $v \in V$.

Because $(v_1, \dots, v_n)$ spans V, $\exists a_1, \dots, a_n \in F$ s.t. $v = a_1 v_1 + \cdots + a_n v_n$.

Suppose there exist another set of $\exists c_1, \dots, c_n \in F$ s.t. $v = c_1 v_1 + \cdots + c_n v_n$.

Then, we have:

$$0 = (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n$$

where $(v_1, \dots, v_n)$ is linearly independent, yielding $a_1 - c_1 = \cdots = a_n c_n = 0$.

"$\Leftarrow$": Given every $v \in V$ can be uniquely written in the form $v = a_1 v_1 + \cdots + a_n v_n$ for some $(a_i)_{i=1}^n$. By definition, $(v_1, \dots, v_n)$ spans V.

To show linear independence, consider $0_V = 0 = a_1 v_1 + \dots a_n v_n$ that is uniquely determined; thus, $a_1 = \cdots = a_n = 0$ must hold. By definition, $(v_1, \dots, v_n)$ is linearly independent. □

2.3 Dimension

Definition. The dimension of a finite-dimensional vector space V is $\#B$, where B is any basis of V.

Theorem. If $\dim V = \dim W$, then $V = W$.

Proof. Every vector in V is a linear combination of a basis $(u_1, \dots, u_n)$ of U that is linearly independent list of length $\dim U = n$. □

Theorem. $(v_1, \dots, v_n)$ is a basis of V iff $\forall v \in V$ can be written uniquely as

$$v = a_1 v_1 + \cdots + a_n v_n$$

Theorem. Given V_1, V_2: subspaces of a finite-dimensional vector space. Then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

Problem. Let U be the subset of $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ consisting of all linear maps $T : \mathbb{R}^3 \to \mathbb{R}^2$ s.t. $\dim \operatorname{null} T > 1$. Prove that U is not a subspace of $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$.

Proof. Consider $T, S \in U$ s.t. $T(x_1, x_2, x_3) = (x_1, 0), S(x_1, x_2, x_3) = (0, x_2)$. Then:

$$\begin{aligned}
&\dim \operatorname{null} T = 2, \quad \dim \operatorname{null} S = 2 \\
&(T+S)(x_1, x_2, x_3) = (x_1, x_2) \quad \Rightarrow \quad \dim \operatorname{range}(T+S) = 2, \quad \dim \operatorname{null}(T+S) = 1 \quad \operatorname{span}(0,0,x_3)
\end{aligned}$$

i.e. $T + S \notin U$ and U is not closed under addition. $\square$

Chapter 3

Linear maps

In set theory, objects are sets.
A map (or a function) $f : S_1 \to S_2$ is a rule assigning $\forall s \in S_1$ some element of S_2, denoted $f(s)$.

$$S_1 \xrightarrow{f} S_2 \Leftrightarrow s \mapsto f(s)$$

3 classes of maps:

- Injective (one-to-one): $\#S_1 \leq \#S_2$
- Surjective (onto): $\#S_1 \geq \#S_2$
- Bijective (one-to-one correspondence): $\#S_1 = \#S_2$

Let V, W be two vector spaces over F.

Definition. A map $T : V \to W$ (viewed as sets) is called a linear map (or a linear transformation) if:

(1) $T(u + v) = T(u) + T(v) \quad \forall u, v \in V$ (additivity)

(2) $T(\lambda \cdot u) = \lambda \cdot T(u) \quad \forall \lambda \in F, u \in V$ (homogeneity)

$F^{m,n}$ is the set of all $m \times n$ matrices with basis consisting of all matrices with a single entry 1 and all other entries 0s and thus of dimension mn.

Theorem. Let $(v_1, \dots, v_n)$ be a basis of V. Let $(w_1, \dots, w_n)$ be a list of elements of W. Then, $\exists$ unique $T : V \to W$ (linear map) s.t.

$$v_i \mapsto w_i \quad \forall i = 1, \dots, n$$

i.e. $T(v_i) = w_i$.

Proof. Since $(v_1, \dots, v_n)$ is the basis of V, then $\forall v \in V : v = c_1 v_1 + \cdots + c_n v_n$ uniquely.

Define $T : V \to W$ s.t.:

$$\begin{aligned} T(v) &= T(c_1v_1 + \cdots + c_nv_n) \\ &= c_1T(v_1) + \cdots + c_nT(v_n) \\ &= c_1w_1 + \cdots + c_nw_n \end{aligned}$$

Note that:

$$\begin{aligned} T(u+v) &= T((c_1v_1 + \cdots + c_nv_n) + (a_1v_1 + \cdots + a_nv_n)) \\ &= T((a_1+c_1)v_1 + \cdots + (a_n+c_n)v_n) \\ &= (a_1+c_1)T(v_1) + \cdots + (a_n+c_n)T(v_n) \\ &= (a_1+c_1)w_1 + \cdots + (a_n+c_n)w_n \\ &= a_1w_1 + \cdots + a_nw_n + c_1w_1 + \dots c_nw_n \\ &= T(u) + T(v) \end{aligned}$$

Similarly, $T(\lambda v) = \lambda T(v)$.

To show uniqueness, combine additivity and homogeneity of any such T:

$$T(v) = T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

This is exactly T we defined. Thus, T is uniquely determined on V. $\square$

Definition. Given V, W: two vector spaces over F, let $\mathcal{L}(V, W)$ be the set of all linear maps $T : V \to W$.

Theorem. $\mathcal{L}(V, W)$ has natural binary operations $+, \cdot$ making it a vector space.

$$\begin{aligned} &T_1, T_2 \in \mathcal{L}(V, W) : (T_1 + T_2)(v) = T_1(v) + T_2(v) \\ &T \in \mathcal{L}(V, W) : (\lambda T)(v) = \lambda T(v) \end{aligned}$$

$\forall v \in V$.

Theorem. Suppose $T : V \to W$. Then $T(0_V) = 0_W$.

Proof. $T(0) = T(0 + 0) = T(0) + T(0)$ i.e. $T(0) = 0$. $\square$

3.1 Null spaces and ranges

Given any linear map $T : V \to W$ (V, W: vector spaces over F), we can define two subspaces in V:

(1) $\text{null}(T) := \{v \in V : T(v) = 0_W\}$

(2) $\text{range}(T) := \{T(v) : v \in V\}$

Injective map

Theorem. $\text{null}(T)$ is a subspace of V.

Proof. Use the criterion for subspace:

(1) $0_V \in \text{null}(T)$ since $T(0_V) = 0_W$

(2) $u, v \in \text{null}(T) : u + v \in \text{null}(T)$, i.e.

$$T(u+v) = T(u) + T(w) = 0_W + 0_W = 0_W$$

(3) $u \in \text{null}(T), \lambda \in F : \lambda u \in \text{null}(T)$, i.e.

$$T(\lambda u) = \lambda T(u) = \lambda 0_W = 0_W$$

□

Definition. A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$.

Theorem. T is injective: $T(u) \neq T(v)$ if $u \neq v \iff \text{null}(T) = \{0_V\}$.

Key: $\forall u, v \in V : Tu = Tv$ implies $Tu - Tv = T(u - v) = 0$ i.e. $u - v \in \text{null}(T) = 0_V$ i.e. $u - v = 0_V$.

Surjective map

Theorem. $\text{range}(T)$ is a subspace of W.

Proof. Use the criterion for subspace:

(1) $0_W \in \text{range}(T)$ since $T(0_V) = 0_W$

(2) $u, v \in \text{range}(T) : u + v \in \text{range}(T)$

(3) $u \in \text{range}(T), \lambda \in F : \lambda u \in \text{range}(T)$

□

Definition. A function $T : V \to W$ is called surjective if $\text{range}(T) = W$.

Fundamental theorem of linear maps

Theorem. Suppose V is a finite-dimensional and $T \in \mathcal{L}(V, W)$. Then, $\text{range}(T)$ is finite-dimensional and

$$\boxed{\dim V = \dim \text{null}(T) + \dim \text{range}(T)}$$

Proof. Let $u_1, \ldots, u_m$ be a basis of $\text{null}(T)$ (i.e. all are linearly independent).

Extend it to $u_1, \ldots, u_m, v_1, \ldots, v_n$ be a basis of V.

Thus, we have $\dim V = m + n$ and $\dim \text{null}(T) = m$.

To show $\dim \text{range}(T) = n$,

Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ spans V, we have $\forall v \in V$:

$$T(v) = T\left(\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j\right) = \sum_{j=1}^{n} b_j T(v_j)$$

since $T(u_i) \in \operatorname{null}(T)$.

Thus, $T(v_1), \ldots, T(v_n)$ span range(T).

Consider:

$$T\left(\sum_{j=1}^{n} b_j v_j\right) = 0_W$$

i.e. $\sum_{j=1}^{n} b_j v_j \in \operatorname{null}(T)$.

Then, we have:

$$\sum_{j=1}^{n} b_j v_j - \sum_{i=1}^{m} a_i u_i = 0$$

iff $a_1, \ldots, a_m, b_1, \ldots, b_n$ are zero since $u_1, \ldots, u_m, v_1, \ldots, v_n$ are basis V and thus linearly independent.

By definition, $T(v_1), \ldots, T(v_n)$ is linearly independent.

Therefore, $T(v_1), \ldots, T(v_n)$ is a basis of range(T), proving $\dim \operatorname{range}(T) = n$.

$\square$

3.2 Matrices

Key: the kth column of $\mathcal{M}(T)$ in $F^{m\times 1}$ is $T(v_k)|_\beta$ for a basis $\beta = (v_1, \dots, v_n)$ of V.

Definition. Given $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$: a basis of V, $w_1, \dots, w_m$: a basis of W.

The matrix of T w.r.t. bases is the $m \times n$ matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by:

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

Theorem. Given $T \in \mathcal{L}(U, V), S \in \mathcal{L}(V, W)$: linear maps, $(u_1, \dots, u_m)$: basis of U, $(v_1, \dots, v_n)$: basis of V, $(w_1, \dots, w_p)$: basis of W. Then:

$$\mathcal{M}(ST, (u_1, \dots, u_m), (w_1, \dots, w_p)) = \mathcal{M}(S, (v_1, \dots, v_n), (w, \dots, w_p))\mathcal{M}(T, (u_1, \dots, u_m), (v_1, \dots, v_n))$$

We denote $A \sim B$ and say that A and B are *similar*.

Theorem. Given $T \in \mathcal{L}(V)$: a linear map, $(u_1, \dots, u_n), (v_1, \dots, v_n)$: bases of V.

Let $A = \mathcal{M}(T, (u_1, \dots, u_n)), B = \mathcal{M}(T, (v_1, \dots, v_n))$, and $C = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$ where $I : V \to V$ is identity operator.

Then

$$A = C^{-1}BC$$

Proof. Note that $T = ITI$ and $C^{-1} = \mathcal{M}(I, v \to u)$.

$$\begin{aligned} A &= \mathcal{M}(T, u \to u) = C^{-1}\mathcal{M}(T, u \to v) \\ &= C^{-1}\mathcal{M}(T, v \to v), \mathcal{M}(I, u \to v) \\ &= C^{-1}BC \end{aligned}$$

□

3.3 Invertibility and isomorphisms

Definition. A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity operator I_V on V and TS equals the identity operator I_W on W.

Theorem. A linear map is invertible iff it is injective and surjective.

Proof.
"$\Rightarrow$": Given T is invertible i.e. $\exists T^{-1} : W \to V$.

To show injective, consider $\forall u, v \in V$ s.t. $Tu - Tv$:

$$u = T^{-1}Tu = T^{-1}Tv = v$$

To show surjective, consider $\forall w \in W$:

$$w = T(T^{-1}w)$$

i.e. $\text{range}(T) = W$. ■
"$\Leftarrow$": Consider a linear map $S : W \to V$ s.t. $T(S(w)) = w \quad \forall w \in W$. Then, $T \circ S = I$ on W.

We show that $S \circ T = I$ on V by:

$$T((S \circ T)v) = (T \circ S)Tv = Tv \quad \forall v \in V$$

□

Lemma. An invertible linear map has a unique inverse.

Proof. (Proof by contradiction) Consider $T \in \mathcal{L}(V, W)$: invertible with two inverses S_1 and S_2.

By definition, we have:

$$S_1 = S_1 I = S_1(TS_2) = (S_1T)S_2 = IS_2 = S_2$$

□

Key: If $T(v_1), \ldots, T(v_n)$ is a basis of W where $T : V \to W$ is a linear map, then T is an isomorphism.

Definition. An isomorphism is an invertible linear map. Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

Theorem. Two finite-dimensional vector spaces over F are isomorphic iff they have the same dimension.

Key: $\dim V = \dim W$ since T must be bijective to be invertible.

Proof.
"$\Rightarrow$": Given V, W and $T : V \to W$ invertible. T is invertible iff it is bijective i.e. both injective and surjective. By definition, $\text{null}T = \{0_V\}$ and $\text{range}T = W$.

Using FTLM, $\dim V = \dim \text{null}T + \dim \text{range}T = 0 + \dim W$. ■
"$\Leftarrow$": Given $\dim V = \dim W$. Define basses $(v_1, \ldots, v_n)$ of V and $(w_1, \ldots, w_n)$ of W. Choose $T \in \mathcal{L}(V, W)$ s.t. $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$.

We have $T(0) = 0$ since both bases are linearly independent and thus $\text{null}(T) = \{0_V\}$ i.e. T is injective.

By deifnition, $\forall w \in W : w = \sum_{i=1}^{n} c_i w_i$ and thus $\text{range}(T) = W$ i.e. T is surjective.

Therefore, T is invertible and is an isomorphism by definition. $\square$

Problem. Given V: finite-dimensional vector space over a field F, $A, B : V \to V$: linear operators. Show that AB and BA has same eigenvalues.

Proof. Let λ be an eigenvalue of AB. Then, $\exists$ some nonzero eigenvector $v \in V$ s.t. $ABv = \lambda v$.

Then, we have:

$$B(ABv) = \lambda Bv = \lambda w = BAw$$

for some nonzero $w = Bv \in V$ i.e. λ is also eigenvalue of BA. $\blacksquare$

Consider $w = Bv = 0$. Then, $ABv = A \cdot 0 = 0 = \lambda v$ and $\lambda = 0$.

Note that since $Bv = 0$ for some nonzero v, the null space of B contains v and is nonzero. By rank-nullity theorem, we have $\dim V = \text{range}B + \text{null}B$ i.e. $\text{rank}B = \dim B < \dim V$.

Thus, we have $\dim BA < \dim V$ i.e. $\exists$ some nonzero $u \in V$ s.t. $BAu = 0$ and therefore $\lambda = 0$ is eigenvalue of BA as well.

$\square$

Products and quotients of vector spaces

Theorem. If V, W: finite-dimensional, then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = \dim V \cdot \dim W$$

3.4 Duality

Dual space and dual map

Definition. A linear functional ϕ on V is a linear map from V to F, i.e. an element of $\mathcal{L}(V, F)$.

Definition. The dual space V' of V is the vector space of all linear functionals on V.

$$V' = \mathcal{L}(V, F) = \{\phi \mid \phi : V \to F\}$$

Definition. For $U \subset V$, the annihilator U^0 is

$$U^0 = \{\phi \in V' : \phi(u) = 0 \quad \forall u \in U\}$$

Theorem. If V is finite-dimensional, then V' is also finite-dimensional and

$$\dim V = \dim V'$$

Proof. $\dim V' = \dim \mathcal{L}(V, F) = (\dim V)(\dim F) = \dim V$. □

Definition. Given $(v_1, \ldots, v_n)$ is a basis of V, the dual basis $(\phi_1, \ldots, \phi_n)$ of V is a set of linear functionals

$$\phi_i(v_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Theorem. The dual basis of basis of V is a basis of dual space V'.

Proof. Given $(v_1, \ldots, v_n)$: basis of V, $(\phi_1, \ldots, \phi_n)$: dual basis.

Consider $c_1\phi_1 + \cdots + c_n\phi_n = 0$. Then, we have:

$$(c_1\phi_1 + \cdots + c_n\phi_n)(v_k) = c_k\phi_k(v_k) = c_k = 0 \quad \forall k = 1, \ldots, n$$

i.e. $(v_1, \ldots, v_n)$ is linearly independent.

Note that $\dim V' = \dim \mathcal{L}(V, F) = \dim V \dim F = \dim V = n$ and dual basis has n elements.

Therefore, $(\phi_1, \ldots, \phi_n)$ is a basis of V'. □

Definition. The dual map T' of T is a linear map in $\mathcal{L}(W', V')$ defined by

$$T'(\phi) = \phi \circ T \in V' \quad \forall \phi \in W'$$

Example. A linear map $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ s.t. $Dp = p'$. A dual map D' s.t. $D'\phi = \phi D$. Consider some ϕ s.t. $\phi(p) = p(3)$:

$$(D'\phi)(p) = (\phi D)p = \phi(Dp) = \phi p' = p'(3)$$

Theorem (Algebraic properties of dual maps)**.** Given $T \in \mathcal{L}(V, W)$. Then:

1. $(S + T)' = S' + T' \quad \forall S \in \mathcal{L}(V, W)$

2. $(\lambda T)' = \lambda T' \quad \forall \lambda \in F$

3. $(ST)' = T'S' \quad \forall S \in \mathcal{L}(W, U)$ $\quad ST : V \to U, (ST)' : U' \to V'$

Proof. (Part 3) Consider any $\phi \in U'$

$$\begin{aligned}(S \circ T)' \circ \phi &= \phi(S \circ T) = (\phi \circ S) \circ T \quad \phi \circ S : W \to F \\ &= T' \circ (\phi \circ S) \\ &= T' \circ (S' \circ \phi) \\ &= (T' \circ S') \circ \phi\end{aligned}$$

□

Theorem. Given $U \subset V$. Then $U^0 \subset V'$.

Proof. Use criterion of a subspace:

(1) A zero linear functional $0 \in U^0$

(2) $\forall \phi, \psi \in U^0$:

$$(\phi + \psi)(u) = \phi(u) + \psi(u) = 0 \quad \forall u \in U$$

i.e. $\phi + \psi \in U^0$

(3) closed under multiplication

□

Theorem. V: vector space over F, $U \subset V$. Then

$$\dim U^0 = \dim V - \dim U$$

Proof. Consider an inclusion map $i : U \to V$ s.t. $i(u) = u$ if $u \in V$. Then a dual map $i' : V' \to U'$ is a linear map s.t. $\forall \phi \in V' : i'\phi = \phi i$.

By FTLM, $\dim V' = \dim \operatorname{null} i' + \dim \operatorname{range} i' = \dim U^0 + \dim V$.

$$\begin{aligned}\operatorname{null} i' &= \{\phi \in V' : i'\phi = 0 = \phi i\} \quad U^0 = \{\phi \in V' : \phi(u) = 0 \quad \forall u \in U\} \text{ and } \phi i : U \to F \\ &= U^0\end{aligned}$$

□

Theorem. Given V: finite-dimensional, $U \subset V$. Then:

(a) $U^0 = \{0\} \iff U = V$

(b) $U^0 = V' \iff U = \{0\}$

Proof. (a) $U^0 = \{0\} \iff \dim V = \dim U \iff V = U$ ■

(b) $U^0 = V' \iff \dim U^0 = \dim V - \dim U = \dim V' - \dim U \iff \dim U = 0 \iff U = \{0\}$

□

Theorem. Given V, W: finite-dimensional, $T \in \mathcal{L}(V, W)$. Then:

(a) $\text{null} T' = (\text{range} T)^0$

(b) $\dim \text{null} T' = \dim \text{null} T + \dim W - \dim V$

Proof. (a) Note that $T' : W' \to V'$.

Let $\phi \in T'$. Then $T'(\phi) = 0 = \phi \circ T$ i.e.

$$(\phi \circ T)(v) = \phi(Tv) = 0 \quad v \in V$$

i.e. $\phi \in (\text{range} T)^0$.

□

Chapter 4

Polynomials

Definition. A polynomial $f(t)$ in $\mathcal{P}(F)$ splits over F if there are scalars $c, a_1, \dots, a_n \in F$ (not necessarily distinct) s.t.

$$f(t) = c(t - a_1)(t - a_2)\dots(t - a_n)$$

where F denotes $\mathbb{R}$ or $\mathbb{C}$.

> Suppose $z = a + bi \in \mathbb{C}$, where $a, b \in \mathbb{R}$.
> The **complex conjugate** of $z \in \mathbb{C}$ is $\bar{z} = a - bi = \text{Re}z - (\text{Im}z)i$.
> The absolute value of $z \in \mathbb{C}$ is $|z| = \sqrt{a^2 + b^2} = \sqrt{(\text{Re}z)^2 + (\text{Im}z)^2}$.
> Note that $z\bar{z} = a^2 - b^2i^2 = a^2 + b^2 = |z|^2$

Theorem. Suppose m is a positive integer and $p \in \mathcal{P}(F)$ is a polynomial of degree m. Fix $\lambda \in F$. Then $p(\lambda) = 0$ iff there exists a polynomial $q \in \mathcal{P}(F)$ of degree $m - 1$ s.t.

$$p(z) = (z - \lambda)q(z) \quad \forall z \in F$$

Proof.
"$\Rightarrow$": Given $p(\lambda) = 0$. Consider $a_0, a_1, \dots, a_m \in F$ s.t. $p(z) = a_0 + a_1 z + \dots + a_m z^m \quad \forall z \in F$.

Then, we have:

$$\begin{aligned} p(z) &= p(z) - p(\lambda) \\ &= a_1(z - \lambda) + \dots + a_m(z^m - \lambda^m) \quad \forall z \in F \end{aligned}$$

Note that we have:

$$z^k - \lambda^k = (z - \lambda)\sum_{j=1}^{k} \lambda^{j-1} z^{k-j} \quad \forall k = 1, \dots, m$$

■

"$\Leftarrow$": Given $\exists q(z) \in \mathcal{P}(F) : p(z) = (z - \lambda)q(z) \quad \forall z \in F$.

Then, we have $p(\lambda) = (\lambda - \lambda)q(\lambda) = 0$. □

Theorem (Fundamental theorem of algebra (FTA), first version)**.** Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Proof. By De Moivre's theorem, we have:

$$\left(r^{1/k}\left(\cos\frac{\theta}{k}+i\sin\frac{\theta}{k}\right)\right)^k = w$$

i.e. every complex number has a kth root. □

Theorem (Fundamental theorem of algebra (FTA), second version)**.** If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization

$$p(z) = c(z-\lambda_1)\dots(z-\lambda_m)$$

where $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$.

Consider $T \in \mathcal{L}(V), p \in \mathcal{P}(F)$.
Note that $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$.
$p(T) : V \to V$ is a linear map defined as:

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n$$

Chapter 5

Eigenvalues and eigenvectors

5.1 Invariant subspaces

Definition (Operator)**.** A linear map from a vector space to itself is called an operator.

Definition (Invariant subspace)**.** Given $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $Tu \in U \quad \forall u \in U$.

Thus, U is invariant under T if $T\big|_U : U \to V$ is an operator on U.

Definition (Eigenvalue)**.** Given $T \in \mathcal{L}(V)$. A number $\lambda \in F$ is called eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

Theorem. Given V: finite-dimensional, $T \in \mathcal{L}(V)$, $\lambda \in F$. Then the following are equivalent:

(a) λ is an eigenvalue of T

(b) $T - \lambda I$ is not injective

(c) $T - \lambda I$ is not surjective

(d) $T - \lambda I$ is not invertible

Definition (Eigenvector)**.** Given $T \in \mathcal{L}(V)$, $\lambda \in F$: eigenvalue of T. A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Theorem. Given $T \in \mathcal{L}(V)$. Every list of eigenvectors of T corresponding to distinct eigenvalues of T are linearly independent.

Proof. We proceed by contradiction.

Assume there exist a smallest positive integer m such that there exists a linearly dependent list $v_1, \ldots, v_m$ of eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ of T.

Note that $m \geq 2$ must hold to form linearly dependent list.

Then, there exist $a_1, \ldots, a_m \in F$ all nonzero (because of minimiality of m):

$$\begin{aligned} a_1 v_1 + \cdots + a_m v_m &= 0 \quad \text{multiply by } T - \lambda_m I \\ &= a_1(\lambda_1 - \lambda_m)v_1 + \cdots + a_m(\lambda_{m-1} - \lambda_m)v_{m-1} \end{aligned}$$

where $\lambda_1, \ldots, \lambda_{m-1}$ are distinct and none of the coefficients above are zero.

Thus, $v_1, \ldots, v_{m-1}$ are linearly dependent. Contradiction. □

Theorem. Each operator on finite-dim V has at most $\dim V$ distinct eigenvalues.

Theorem. Given $T \in \mathcal{L}(V), p \in \mathcal{P}(F)$. null $p(T)$ and range $p(T)$ are invariant under T.

Proof. Consider $u \in \text{null } p(T)$. Then $p(T)u = 0$. Thus, we have:

$$p(T)(Tu) = T(p(T)u) = T0 = 0$$

i.e. $Tu \in \text{null } p(T)$. ■

Consider $u \in \text{range } p(T)$. Then there exists $v \in V$ such that $p(T)v = u$. Thus, we have:

$$Tu = T(p(T)v) = p(T)(Tv)$$

where $Tv \in V$ by definition i.e. $Tu \in \text{range } p(T)$. □

5.2 Minimal polynomial

5.2.1 Existence of eigenvalues on complex vector spaces

Theorem. Every operator on a finite-dim nonzero complex vector space V has an eigenvalue.

Proof. Consider any $T \in \mathcal{L}(V)$, where $\dim V = n$. Choose $v \in V : v \neq 0$. Then, we have a list of vectors $v, Tv, T^2v, \ldots, T^nv$ of length $n+1$ that is not linearly independent.

Thus, there exist a nonconstant polynomial p of smallest degree such that $p(T)v = 0$.

By the first version of FTA, there exist $\lambda \in \mathbb{C}$ such that $p(\lambda) = 0$.

Then, there exists a polynomial $q \in \mathcal{P}(\mathbb{C})$ such that $p(z) = (z - \lambda)q(z) \quad \forall z \in \mathbb{C}$.

Then, we have: $0 = p(T)v = (T - \lambda I)(q(T)v)$.

Since q has smaller degree than p, we know that $q(T)v \neq 0$.

Thus, we have that λ is an eigenvalue of T with eigenvector $q(T)v$. □

5.2.2 Eigenvalues and minimal polynomial

Key: find smallest positive integer m s.t.

$$c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} = -T^m$$

has a solution $c_0, \ldots, c_{m-1} \in F$.

Definition. Given V: finite-dimensional, $T \in \mathcal{L}(V)$. Then the minimal polynomial of T is the *unique* monic polynomial $p \in \mathcal{P}(F)$ of smallest degree s.t. $p(T) = 0$.

Quaternions by Hamilton By Frobenius theorem, the algebra $\mathbb{H}$ is one of only two finite-dimensional division rings containing a proper subring isomorphic to the real numbers; the other is complex numbers $\mathbb{C}$.

5.3 Upper-triangular matrices

Theorem. Given $T \in \mathcal{L}(V)$, V has a basis w.r.t. which T has an upper triangular matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Then

$$(T - \lambda_1 I) \dots (T - \lambda_n I) = 0$$

5.4 Diagonalizable operators

Definition. An operator T on V is called diagonalizable if the operator has a diagonal matrix $\mathcal{M}(T)\big|_\beta$ w.r.t. to some basis β of V.

Theorem (Conditions for diagonalizability)**.** Given V: finite-dimensional, $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a) T is diagonalizable

(b) V has a basis consisting of eigenvectors of T

(c) $V = \oplus_{i=1}^m E(\lambda_i, T)$

(d) $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$

Proof. An operator $T \in \mathcal{L}(V)$ has a diagonal matrix

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{bmatrix}$$

w.r.t. basis $v_1, \dots, v_n$ iff $Tv_k = \lambda_k v_k$ for all $k \in [n]$. □

Matrix limits and Markov chains

Theorem (Enough eigenvalues implies diagonalizability)**.** Given V: finite-dimensional. If $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues, then T is diagonalizable.

Theorem. Given V: finite-dimensional, $T \in \mathcal{L}(V)$. T is diagonalizable iff the minimal polynomial of T equals $(z - \lambda_1) \dots (z - \lambda_m)$ for some list of distinct $\lambda_1, \dots, \lambda_m \in F$.

Proof. "$\Rightarrow$": Given T is diagonalizable. Then, there exists a basis $(v_1, \dots, v_n)$ of V consisting of eigenvectors of T.

Let $\lambda_1, \dots, \lambda_m$ be distinct eigenvalues of T. Then, for each v_j there exists λ_k s.t. $(T - \lambda_k I)v_j = 0$.

Thus, we have $(T - \lambda_1 I) \dots (T - \lambda_m I)v_j = 0$ implying $(z - \lambda_1) \dots (z - \lambda_m)$ is a minimal polynomial. ■

"$\Leftarrow$": Given $p(T) = \prod_{i=1}^m (z - \lambda_i)$. Then, we have $(T - \lambda_1 I) \dots (T - \lambda_m I) = 0$.

We show by induction on m.

Base case $m = 1$: $T - \lambda_1 I = 0$ i.e. $T = \lambda_1 I$ and thus diagonalizable.

Inductive step. range$(T - \lambda_m I)$ is invariant under T. Thus, T restricted to range$(T - \lambda_m I)$ is an operator on range$(T - \lambda_m I)$.

Consider any $u \in \text{range}(T - \lambda_m I)$. Then, we have $u = (T - \lambda_m I)v$ for some $v \in V$ and:

$$(T - \lambda_1 I) \dots (T - \lambda_{m-1} I)u = (T - \lambda_1 I) \dots (T - \lambda_m I)v = 0$$

$\square$

Minimal polynomial of linear operator $T \in \mathcal{L}(V)$ with m distinct eigenvalues: $p(z) = \prod_{i=1}^{m} (z - \lambda_i)^{e_i}$ where e_i is the size of largest Jordan block for λ_i (or equivalently smallest k : null $(T - \lambda_i I)^k =$ null $(T - \lambda_i I)^{k+1}$). Characteristic polynomial: $\chi(z) = \prod_{i=1}^{m} (z - \lambda_i)^{d_i}$ where $d_i = \dim G(\lambda_i, T)$ is an algebraic multiplicity.

5.5 Commuting operators

Chapter 6

Inner product spaces

Definition. An inner product on V is a function that takes each ordered pair $\langle u, v\rangle$ of elements of V to a number $\langle u, v\rangle \in F$ and has the following properties:

- positivity: $\langle v, v\rangle \geq 0 \quad \forall v \in V$
- definiteness: $\langle v, v\rangle = 0$ iff $v = 0$
- $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \quad \forall u, v, w \in V$
- homogeneity in first slot: $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle \quad \forall \lambda \in F, \forall u, v \in V$
- conjugate symmetry: $\langle u, v\rangle = \overline{\langle v, u\rangle} \quad \forall u, v \in V$

Definition. An inner product space is a vector space along with an inner product on that vector space.

Theorem (Basic properties of inner product)**.**

(a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v\rangle$ is a linear map from V to F i.e. a functional.

(b) $\langle 0, v\rangle = 0 = \langle v, 0\rangle \quad \forall v \in V$

(c) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle \quad \forall u, v, w \in V$

(d) $\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \bar{\lambda}\overline{\langle v, u\rangle} = \bar{\lambda}\langle u, v\rangle \quad \forall \lambda \in F, \forall u, v \in V$

Definition. For $v \in V$, the norm of v, denoted by $\|v\|$, is defined by $\|v\| = \sqrt{\langle v, v\rangle}$.

Theorem. Given $v \in V$.

(a) $\|v\| = 0$ iff $v = 0$

(b) $\|\lambda v\| = |\lambda|\|v\| \quad \forall \lambda \in F$

Proof. (b). Given $\lambda \in F$, we have:

$$\|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle = \lambda\langle v, \lambda v\rangle = \lambda\bar{\lambda}\langle v, v\rangle = |\lambda|^2\|v\|^2$$

□

Definition. Two vectors $u, v \in V$ are called orthogonal if $\langle u, v \rangle = 0$.

Theorem. Given $u, v \in V$. If u and v are orthogonal, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Proof. $\|u+v\|^2 = \langle u+v, u+v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle = \|u\|^2 + \|v\|^2$. □

Theorem (Orthogonal decomposition)**.** Given $u, v \in V : v \neq 0$. Set $c = \frac{\langle u,v \rangle}{\|v\|^2}$ and $w = u - \frac{\langle u,v \rangle}{\|v\|^2} v$. Then, we have:

$$u = cv + w, \langle w, v \rangle = 0$$

Theorem (Cauchy-Schwarz inequality)**.** Given $u, v \in V$. Then $|\langle u, v \rangle| \leq \|u\| \|v\|$.

This inequality is equality iff one of u, v is a scalar multiple of the other.

Proof. Note that if $v = 0$ the inequality holds trivially.

Consider $v \neq 0$. Consider orthogonal decomposition:

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + w$$

s.t. $w \perp v$.

Then, we have:

$$\begin{aligned} \|u\|^2 &= \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\ &= \frac{|\langle u, v \rangle|^2}{\|v\|^4} \|v\|^2 + \|w\|^2 \geq \frac{|\langle u, v \rangle|^2}{\|v\|^2} \end{aligned}$$

□

6.1 Orthonormal bases

Theorem. Given $e_1, \ldots, e_n$: orthonormal list of vectors in V.

$$\begin{aligned} \|a_1 e_1 + \cdots + a_m e_m\|^2 &= \langle a_1 e_1 + \cdots + a_m e_m, a_1 e_1 + \cdots + a_m e_m \rangle \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} a_i \overline{a_j} \langle e_i, e_j \rangle \\ &= \sum_{i=1}^{m} |a_i|^2 \quad a_i \overline{a_i} = |a_i|^2 \\ &= |a_1|^2 + \cdots + |a_m|^2 \quad \forall a_1, \ldots, a_m \in F \end{aligned}$$

Key: it follows from above that any orthonormal list of vectors is linearly independent.

Theorem. Every orthonormal list of vectors in a finite-dim V of length $\dim V$ is ONB of V.

Theorem (Bessel's inequality)**.** Given $e_1, \ldots, e_m$: orthonormal list of vectors in V. If $v \in V$, then:

$$|\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \leq \|v\|^2$$

Proof. Consider any $v \in V$. Define u, w s.t.:

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m + v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m = u + w$$

Then, we have for $k \in [m]$: $\langle w, e_k\rangle = \langle v, e_k\rangle - \langle v, e_k\rangle\langle e_k, e_k\rangle = 0$ i.e. $\langle w, u\rangle = 0$.
Thus, we have:

$$\|v\|^2 = \|u\|^2 + \|w\|^2 \geq \|u\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2$$

$\square$

Given $e_1, \ldots, e_n$: orthonormal list of vectors in V, $u, v \in V$. Then, we have:

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

where $\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$.
Thus, we have:

$$\langle u, v\rangle = \langle u, e_1\rangle\overline{\langle v, e_1\rangle} + \cdots + \langle u, e_n\rangle\overline{\langle v, e_n\rangle} \quad \text{slot-2 conjugate-linear}$$

Theorem. Every finite-dim inner product space has an ONB.

Theorem (Riesz representation theorem)**.** Given V: finite-dim, ϕ: a linear functional on V. Then $\exists$ a *unique* vector $v \in V$ s.t.

$$\phi(u) = \langle u, v \rangle \quad \forall u \in V$$

Proof. Existence. Let $e_1, \ldots, e_n$ be ONB of V. Then:

$$\begin{aligned}
\phi(u) &= \phi(\langle u, e_1 \rangle e_1 + \cdots + \langle u, e_n \rangle e_n) \\
&= \langle u, e_1 \rangle \phi(e_1) + \cdots + \langle u, e_n \rangle \phi(e_n) \\
&= \langle u, \overline{\phi(e_1)} e_1 + \cdots + \overline{\phi(e_n)} e_n \rangle \quad \forall u \in V
\end{aligned}$$

Choose $v = \overline{\phi(e_1)} e_1 + \cdots + \overline{\phi(e_n)} e_n$. ■

Uniqueness. Assume there exists two $v_1, v_2 \in V : \phi(u) = \langle u, v_1 \rangle = \langle u, v_2 \rangle \quad \forall u \in V$.

Then $0 = \langle u, v_1 \rangle - \langle u, v_2 \rangle = \langle u, v_1 - v_2 \rangle$. Thus, we have $v_1 = v_2$.

□

6.2 Orthogonal complements and minimization

6.2.1 Orthogonal complements

Definition. If U is a subset of V, then the *orthogonal complement* of U, denoted by $U^\perp$, is the set if all vectors in V that are orthogonal to every vector in U: $U^\perp = \{v \in V : \langle u, v\rangle = 0 \quad \forall u \in U\}$.

Theorem. $V = U \oplus U^\perp$.

Proof. Consider $v \in V$. Let $e_1, \dots, e_m$ be an ONB of U.

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m + v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_m\rangle e_m$$

□

6.2.2 Minimization

Example. Suppose we want to find a polynomial u with real coefficients and of degree at most 5 that approximates the sine functions as well as possible on the interval $[-\pi, \pi]$ s.t.

$$\min \int_{-\pi}^{\pi} |\sin x - u(x)| dx$$

Let $C[-\pi, \pi]$ denote the real inner product space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} fg$$

Apply GSP to basis of U to get orthonormal basis (ONB).

Chapter 7

Operations on inner product spaces

7.1 Self-adjoint and normal operators

Definition. The adjoint of $T \in \mathcal{L}(V, W)$ is a function $T^* : W \to V$ s.t. $\langle Tv, w\rangle = \langle v, T^*w\rangle\ \forall v \in V, w \in W$.

Theorem. Adjoint of a linear map is a linear map.

Proof. Given $T \in \mathcal{L}(V, W)$. Consider $v \in V, w_1, w_2 \in W$:

$$\begin{aligned}\langle v, T^*(w_1 + w_2)\rangle = \langle Tv, w_1 + w_2\rangle &= \langle Tv, w_1\rangle + \langle Tv, w_2\rangle \\ &= \langle v, T^*w_1\rangle + \langle v, T^*w_2\rangle \\ &= \langle v, T^*w_1 + T^*w_2\rangle\end{aligned}$$

i.e. $T^*(w_1 + w_2) = T^*w_1 + T^*w_2$.

Note that we also have:

$$\begin{aligned}\langle v, \lambda T^*w\rangle = \langle Tv, \lambda w\rangle &= \bar{\lambda}\langle Tv, w\rangle \\ &= \bar{\lambda}\langle v, T^*w\rangle\end{aligned}$$

□

Theorem (Properties of adjoint)**.** Given $T \in \mathcal{L}(V, W)$.

(a) $(S + T)^* = S^* + T^* \quad \forall S \in \mathcal{L}(V, W)$

(b) $(\lambda T)^* = \bar{\lambda}T^* \quad \forall \lambda \in F$

(c) $(T^*)^* = T$

(d) $(ST)^* = T^*S^* \quad \forall S \in \mathcal{L}(W, U)$ where U: finite-dim inner product space over F

(e) $I^* = I$

(f) If T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$

Proof.
(d). Given $S \in \mathcal{L}(W, U)$. Note that $ST : V \to U$. Consider $u \in U$:

$$\langle (ST)v, u \rangle = \langle S(Tv), u \rangle = \langle Tv, S^* u \rangle = \langle v, T^*(S^* u) \rangle$$

$\blacksquare$ $\square$

Theorem (Null space and range of T^*)**.** Given $T \in \mathcal{L}(V, W)$.

(a) null $T^* = (\text{range } T)^\perp$

(b) range $T^* = (\text{null } T)^\perp$

(c) null $T = (\text{range } T^*)^\perp$

(d) range $T = (\text{null } T^*)^\perp$

Proof.
(a). Note that $T^* : W \to V$. Consider $w \in W$ such that $w \in \text{null } T^*$ i.e. $T^* w = 0$:

$$\begin{aligned} \langle Tv, w \rangle &= \langle v, T^* w \rangle \\ &= \langle v, 0 \rangle = 0 \quad \forall v \in V \end{aligned}$$

i.e. $Tv \perp w$ and $w \in (\text{range } T)^\perp$. $\blacksquare$ $\square$

Definition. The conjugate transpose of $m \times n$ matrix A is $n \times m$ matrix A^* obtained by transpose and then taking the complex conjugate of each entry.

Theorem. Matrix of T^* equals conjugate transpose of matrix of T.

$$\mathcal{M}(T^*) = (\mathcal{M}(T))^*$$

s.t. $T \in \mathcal{L}(V, W)$, $e_1, \ldots, e_n$: orthonormal basis of V, $f_1, \ldots, f_m$: orthonormal basis of W. Then $\mathcal{M}(T^*, (f_1, \ldots, f_m), (e_1, \ldots$
is the conjugate transpose of $\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$.

Definition (Hermitian)**.** A self-adjoint matrix is a matrix that is equal to its conjugate transpose matrix.

Theorem (Properties of a Hermitian matrix)**.**

- all eigenvalues are real
- eigenvectors corresponding to distinct eigenvalues are orthogonal

Note that if a Hermitian matrix only contains real entries, it is a symmetric matrix.

7.1.1 Self-adjoint operators

Definition. An operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T = T^*$.

$T \in \mathcal{L}(V)$ is self-adjoint iff $\langle Tv, w\rangle = \langle v, Tw\rangle \quad \forall v, w \in V$.

Theorem. Every eigenvalue of a self-adjoint operator is real.

Proof. Given $T \in \mathcal{L}(V)$ self-adjoint. Consider eigenvalue λ of T with nonzero eigenvector v s.t. $Tv = \lambda v$.

Then, we have:

$$\begin{aligned} \lambda\|v\|^2 &= \langle \lambda v, v\rangle \\ &= \langle Tv, v\rangle \\ &= \langle v, Tv\rangle = \langle v, \lambda v\rangle \\ &= \bar{\lambda}\|v\|^2 \end{aligned}$$

i.e. $\lambda = \bar{\lambda}$ and λ is real. □

7.1.2 Normal operators

Definition. An operator on an inner product space is called *normal* if it commutes with its adjoint, i.e. $T \in \mathcal{L}(V)$ is normal iff $TT^* = T^*T$.

Theorem. $T \in \mathcal{L}(V)$ is normal iff $\|Tv\| = \|T^*v\| \quad \forall v \in V$.

Proof. Note that since T is normal, we have $TT^* - T^*T = 0$.

Then, we have $\quad \forall v \in V$:

$$\begin{aligned} \langle (TT^* - T^*T)v, v\rangle = 0 &\iff \langle TT^*v, v\rangle = \langle T^*Tv, v\rangle \\ &\iff \|T^*v\| = \langle T^*v, T^*v\rangle = \langle Tv, (T^*)^*v\rangle = \langle Tv, Tv\rangle = \|Tv\| \end{aligned}$$

□

Theorem. $T \in \mathcal{L}(V)$: normal.

(a) null T = null T^* and range T = range T^*

(b) V = null $T \oplus$ range T

(c) $T - \lambda I$ is normal for all $\lambda \in F$

(d) $Tv = \lambda v$ iff $T^*v = \bar{\lambda}v \quad \forall v \in V, \lambda \in F$

Proof. (c).

$$\begin{aligned} (T - \lambda I)(T - \lambda I)^* &= (T - \lambda I)(T^* - \bar{\lambda}I) \\ &= TT^* - \bar{\lambda}T - \lambda T^* + |\lambda|^2 I \\ &= T^*T - \bar{\lambda}T - \lambda T^* + |\lambda|^2 I \\ &= (T^* - \bar{\lambda}I)(T - \lambda I) \\ &= (T - \lambda I)^*(T - \lambda I) \end{aligned}$$

■

(d). $\|(T - \lambda I)v\| = \|(T - \lambda I)^*v\| = (T^* - \bar{\lambda}I)v = 0$. □

Theorem. Eigenvectors of normal operator $T \in \mathcal{L}(V)$ corresponding to distinct eigenvalues are orthogonal.

Proof. Consider distinct eigenvalues $\alpha \neq \beta$ of T with corresponding eigenvectors u, v i.e. $Tu = \alpha u, Tv = \beta v$.

Note that $T^* v = \bar{\beta} v$.

Then, we have:

$$\begin{aligned}(\alpha - \beta)\langle u, v\rangle &= \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle \\ &= \langle Tu, v\rangle - \langle u, T^* v\rangle = 0\end{aligned}$$

i.e. $\langle u, v\rangle = 0$ must hold. $\square$

7.2 Spectral theorem

Theorem (Real spectral theorem)**.** Given $F = \mathbb{R}$, $T \in \mathcal{L}(V)$. The following are equivalent.

(a) T is self-adjoint

(b) T has a diagonal matrix w.r.t. some ONB of V

(c) V has an ONB consisting of eigenvectors of T

Proof. (a) $\Rightarrow$ (b). Given T is self-adjoint. $\square$

7.3 Positive operators

7.4 Isometries, unitary operators, and matrix factorization

7.5 SVD

7.5.1 Approximation by linear maps with lower-dimensional range

Chapter 8

Operators on complex vector spaces

8.1 Generalized eigenvectors and nilpolent operators

Definition. Given $T \in \mathcal{L}(V)$, λ: eigenvalue of T. A vector $v \in V$ is called a *generalized eigenvector* of T corresponding to λ if $v \neq 0$ and

$$(T - \lambda I)^k v = 0$$

for some positive integer k.

A nonzero vector $v \in V$ is a generalized eigenvector of T corresponding to λ iff $(T - \lambda I)^{\dim V} v = 0$

Lemma. $\text{null}T^0 = \{0\} \subset \text{null}T \subset \text{null}T^2 \subset \cdots \subset \text{null}T^k$.

Proof. $v \in \text{null}T^k \Leftrightarrow T^k v = 0$. Then, we have $T(T^k v) = 0 = T^{k+1}v \Leftrightarrow v \in \text{null}T^{k+1}$. □

Define $d(k) = \dim(\text{null}T^k) \quad \forall k = 0, 1, 2 \ldots.$

Lemma ($d(k)$ stabilizes)**.** If $\text{null}T^m = \text{null}T^{m+1}$, then $\text{null}T^{m+j} = \text{null}T^m \quad \forall j = 1, 2, \ldots$

Proof. Equivalent to $\text{null}T^{m+i} \subset \text{null}T^{m+i+1} \quad \forall i = 0, 1, 2, \ldots$
Let $v \in \text{null}T^{m+i+1} \Leftrightarrow T^{m+i+1}v = 0 \Leftrightarrow T^{m+1}T^i v = 0$.
Denote $w = T^i v$.
By our assumption, $w \in \text{null}T^m \Leftrightarrow T^m w = 0 \Leftrightarrow T^{m+i}v = 0$. □

Lemma ($d(k)$ stabilizes at $k = n$ at the latest)**.** Let $n = \dim V$. Then $\text{null}T^n = \text{null}T^{n+1}$.

Theorem. V: finite-dimensional vector space over $\mathbb{C}$, $T : V \to V$.

$$V = \text{null}T^n \oplus \text{range}T^n$$

Remark. If $T : V \to V$ is diagonalizable, $V = \text{null}T \oplus \text{range}T$.

Example. Given $T : V \to V$ isomorphism i.e. an invertible linear map.
Then $d(k) = \dim \text{null}(T^k) = \dim\{0_V\} = 0$.

Nilpolent operators

Definition. An operator is called *nilpolent* if some power of it is equal 0.

Key: if an operator T is nilpotent, then repeatedly applying it to the vector $v \in V$ eventually flattens it to 0. There exists a power $k \in \mathbb{R}$ s.t. T^k will flatten *any* vector $v \in V$.

Theorem. Suppose $T \in \mathcal{L}(V)$ is nilpolent. Then $T^{\dim V} = 0$.

Proof. Since T is nilpolent, $\exists$ some power k s.t. $T^k = 0$. Thus, we have $\text{null}T^k = V = \text{null}T^{\dim V}$ and $T^{\dim V} = 0$. □

Theorem. Suppose $T \in \mathcal{L}(V)$.

(1) If T is nilpolent, then 0 is an eigenvalue of T and T has no other eigenvalues.

(2) If $F = \mathbb{C}$ and 0 is the only eigenvalue of T, then T is nilpolent.

Proof. (1) By definition, $\exists$ some power m s.t. $T^m = 0$. Then, T maps at least one nonzero vector $T^{m-1}v = 0$.

Thus, T is not injective and $\text{null}T \neq \{0_V\}$.

Therefore, $\exists$ some nonzero $v \in \text{null}T \subseteq V$ s.t. $Tv = 0 = \lambda v$ where $\lambda = 0$.

Assume λ' is nonzero eigenvalue of T. Then, we have $Tv = \lambda' v$ for some nonzero $v \in V$.

Thus, we have $\lambda'^m v = T^m v = 0$. Then, $\lambda' = 0$. ■

(2) The minimal polynomial of T is $(z-0)^m = z^m$.

Thus, we have $T^m = 0$, and T is nilpolent by definition. ■

□

Theorem. $T \in \mathcal{L}(V)$ is nilpolent iff $\exists$ a basis β of V s.t.

$$\mathcal{M}(T)\big|_\beta = \begin{bmatrix} 0 & & * \\ & \dots & \\ 0 & \dots & 0 \end{bmatrix}$$

where all entries on and below diagonal are equal 0.

Proof. "$\Leftarrow$": Note that $T^{\dim V} = 0$. Therefore, T is nilpolent by definition.

"$\Rightarrow$": Note that minimal polynomial of T is z^m and $T^m = 0$ for some positive integer m.

□

8.2 Generalized eigenspace decomposition

Definition. Given $T \in \mathcal{L}(V)$, $\lambda \in F$. The *generalized eigenspace* of T corresponding to λ is

$$G(\lambda, T) = \{v \in V : (T - \lambda I)^k v = 0 \text{ for some } k \in \mathbb{Z}_+\}$$

i.e. $G(\lambda, T)$ is a set of generalized eigenvectors of T corresponding to λ (includes zero).

Key: $(T - \lambda I)^0 = I = \{0\} \subset (T - \lambda I) \subset \cdots \subset (T - \lambda I)^k$

Theorem.

$$V = \oplus_{i=1}^{m} G(\lambda_i, T) \tag{8.1}$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of T (without repetitions).

Proof. First, we show that V has a basis of generalized eigenvectors, so V is a sum of $G(\lambda_i, T)$.

Note that we have:

$$V = \text{null}(T - \lambda_i I)^n \oplus \text{range}(T - \lambda_i I)^n$$

s.t. $(T - \lambda_i I)^{2n} w = 0$ i.e. $w \in \text{null}(T - \lambda_i I)^{2n} = \text{null}(T - \lambda_i I)$.

Thus, we have $(T - \lambda_i I)^n w = 0 = v$.

Therefore, $\text{null}(T - \lambda_i I)^n \cap \text{range}(T - \lambda_i I)^n = \{0\}$.

We show this by induction on n.

<u>Base case</u>. $n = 1$.

<u>Inductive step</u>. Suppose we have proved it on the case $\dim V < n$.

Consider $\lambda_1, \ldots, \lambda_m$: eigenvalues. Fix λ_m:

$$\begin{aligned} V &= \text{null}(T - \lambda_m I)^n \oplus \text{range}(T - \lambda_m I)^n \\ &= G(\lambda_m, I) \oplus V \end{aligned}$$

Lemma. $G(\lambda_i, T)$ is T-invariant and $\text{range}(T - \lambda_i I)^n$ is T-invariant because T commutes with $(T - \lambda_i I)^n$.

■

Proposition. $v \in V$ cannot be a generalized eigenvector with two distinct eigenvalues.

Proof. Suppose $(T - \lambda I)^n = 0, (T - \mu I)^m = 0$ but $(T - \mu I)^{m-1} v \neq 0$ for some $m \leq n$.

Then $((\mu I - \lambda I)(T - \mu I))^n v = 0$.

Use binomial theorem: $(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$. Then, we have:

$$\sum_{i=0}^{n} (T - \mu I)^m (\mu I - \lambda I)^{n-i} (T - \mu I)^i v = 0$$

□

Lemma. The sum $G(\lambda_i, T)$ for $i = 1, \ldots, m$ is a direct sum.

□

Characteristic polynomial

Consider $q(T)u = q(T)v_1 + \cdots + q(T)v_m$.

We know $(T - \lambda_i I)^n v_i = 0$. Moreover, $(T - \lambda_i I)^{d_i} v_i = 0$ for $d_i = \dim G(\lambda_i, T)$.

$$q(T) = \prod_{i=1}^{m} (T - \lambda_i I)^{d_i} u = 0 \quad \forall u \in V$$

Definition. The characteristic polynomial is:

$$q(z) = \prod_{i=1}^{m} (z - \lambda_i)^{d_i}$$

Theorem. $q(T) = 0$.

Theorem. Given $T \in \mathcal{L}(V)$, $\lambda \in F$. Then $G(\lambda, T) = \operatorname{null}(T - \lambda I)^{\dim V}$.

Corollary. The minimal polynomial $p(z)$ divides the characteristic polynomial $q(z)$.

Cayley-Hamilton theorem Cayley-Hamilton theorem allows A^n to be expressed as a linear combination of lower matrix powers of A

$$A^n = -c_{n-1}A^{n-1} - \cdots - c_1 A - c_0 I_n$$

When the ring is a field, the Cayley-Hamilton theorem is equivalent to the statement that the minimal polynomial of a square matrix divides its characteristic polynomial.

8.3 Jordan form, square roots of operators

Definition. Given $T \in \mathcal{L}(V)$. A basis β of V is called a **Jordan basis** for T if w.r.t. this basis, T has a block diagonal matrix:

$$\mathcal{M}\big|_{\beta} = \begin{bmatrix} A_1 & & 0 \\ & \dots & \\ 0 & & A_p \end{bmatrix}, \text{ where } A_k = \begin{bmatrix} \lambda_k & 1 & & \dots \\ & & \dots & \\ & & \dots & 1 \\ 0 & & \dots & \lambda_k \end{bmatrix} \quad \forall k \in [p]$$

Theorem. Given $T \in \mathcal{L}(V)$ is nilpolent. There exists a basis of V that is Jordan basis for T.

Proof. We proceed by induction on $\dim V$.

Base case. $\dim V = 1$.

Inductive step. Consider $\dim V > 1$. Let m be the smallest positive integer s.t. $T^m = 0$. Thus, there exists $u \in V$ s.t. $T^{m-1}u \neq 0$. Let $U = \operatorname{span}(u, Tu, \dots, T^{m-1}u)$.

Note that the list of vectors $u, Tu, \dots, T^{m-1}u$ is linearly independent.

If $U = V$, then writing this in reverse order gives a Jordan basis for T.

Consider $U \neq V$. Note that U is invariant under T i.e. $Tu \in U \quad \forall u \in U$. By induction hypothesis, there exists a basis of U that is a Jordan basis for $T\big|_U$.

□

Define **Jordan chain** for kth vector v_k of basis of V as $\{T^{m_k}v_k, T^{m_k-1}v_k, \dots, Tv_k, v_k\}$. Note that $T \in \mathcal{L}(V)$ operates on each Jordan chain independently.

Note that each basis vector lives in one Jordan chain.

There is a finite list of vectors $v_1, \dots, v_n \in V$ s.t. there is a basis of V consisting of $(T^j v_k)$ $k = 1, \dots, n, j = m_k \geq \dots \geq 0$ s.t. $T^{m_k}v_k \neq 0$. We define m_k as the max power that sends any vector $v \in V$ to 0 in Jordan chain k.

T has $(n \times n) \times (n \times n)$ block diagonal matrix w.r.t. to this basis.

8.4 Trace

A connection between matrices and operators

Chapter 9

Multilinear algebra and determinants

A negative determinant inverts/flips orientation of the space

9.1 Bilinear forms and quadratic forms

Definition. A *bilinear form* on V is a function $\beta : V \times V \to F$ s.t.

$$
\begin{aligned}
v &\to \beta(v, u) \\
v &\to \beta(u, v)
\end{aligned}
$$

are both linear functionals on V for every $u \in V$.

The set of bilinear forms on V is denoted by $V^{(2)}$.

Definition. Suppose β is a bilinear form on V and $e_1, \dots, e_n$ is a basis of V. The *matrix* of β w.r.t. to this basis is the $n \times n$ matrix $\mathcal{M}(\beta)$ whose entry $\mathcal{M}(\beta)_{j,k}$ in row j, column k is given by

$$\mathcal{M}(\beta)_{j,k} = \beta(e_j, e_k)$$

Symmetric bilinear forms

Definition. A bilinear form $\rho \in V^{(2)}$ is called *symmetric* if

$$\rho(u, w) = \rho(w, u) \quad \forall u, w \in V$$

Theorem (Symmetric bilinear forms are diagonalizable)**.** Suppose $\rho \in V^{(2)}$. Then the following are equivalent.

(a) ρ is a symmetric bilinear form on V.

(b) $\mathcal{M}(\rho, (e_1, \dots, e_n))$ is a symmetric matrix for every basis $e_1, \dots, e_n$ of V.

(c) $\mathcal{M}(\rho, (e_1, \dots, e_n))$ is a symmetric matrix for some basis $e_1, \dots, e_n$ of V.

(d) $\mathcal{M}(\rho, (e_1, \dots, e_n))$ is a diagonal matrix for some basis $e_1, \dots, e_n$ of V.

Definition. A bilinear form $\alpha \in V^{(2)}$ is called *alternating* if

$$\alpha(v, v) = 0 \quad \forall v \in V$$

Theorem. A bilinear form α on V is alternating iff

$$\alpha(u, w) = -\alpha(w, u) \quad \forall u, w \in V$$

Theorem.

$$V^{(2)} = V^{(2)}_{\text{sym}} \oplus V^{(2)}_{\text{alt}}$$

Quadratic forms

Definition. Given β: a bilinear form on V. Define a function $q_\beta : V \to F$ by $q_\beta(v) = \beta(v, v)$.

A function $q : V \to F$ is called a *quadratic form* on V if there exists a bilinear form β on V s.t. $q = q_\beta$.

9.2 Alternating multilinear forms

9.3 Determinant

9.3.1 Determinants of order 2

9.3.2 Determinants of order n

Theorem (Hadamard's inequality)**.** Suppose A is $n \times n$ matrix. Let $v_1, \ldots, v_n$ denote the columns of A. Then

$$|\det A| \leq \prod_{k=1}^{n} \|v_k\|$$

9.4 Tensor products

Chapter 10

Inner product spaces

10.1 Bilinear and quadratic forms

10.1.1 Einstein's special theory of relativity

10.1.2 Rayleigh's quotient

We use Rayleigh's principle to find the eigenvalues of matrix A.

Given a symmetric, positive definite matrix A we define:

$$R(x) = \frac{x^T A x}{x^T x}$$

Minimax theorem by Von Neumann